Complexity, Combinatorial Positivity, and Newton Polytopes

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Based on joint work with:

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Poorly understood issue: Why are do some decision problems have fast algorithms and others seem to need costly search?

Multiplication is easy:

```
90912135295978188784406583026004374858926083103
28358720428512168960411528640933367824950788367
956756806141 x 814385925911004526572780912628442
93358778990021676278832009141724293243601330041
16702003240828777970252499
```

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Factoring seems hard. RSA $30,000 challenge:

74037563479561712828046796097429573142593188889
23128908493623263897276503402826627689199641962
51178439958943305021275853701189680982867331732
73108930900552505116877063299072396380786710086
096962537934650563796359

Solved in 2012.

Complexity has long connections of combinatorics, but mainly *graph theory* and *optimization*. We'd like to propose a paradigm for *algebraic* combinatorics to connect to complexity.

Computational Complexity Theory II

Complexity has long connections of combinatorics, but mainly *graph theory* and *optimization*. We'd like to propose a paradigm for *algebraic* combinatorics to connect to complexity.

$\therefore$ I now give a brief summary of complexity theory:

- NP: LP ($\exists x \geq 0$, Ax=b?)
- coNP: Primes
- P: LP and Primes!
- NP-complete: Graph coloring

Famous theoretical computer science problems relevant to us:

- P $\stackrel{?}{=}$ NP
- NP $\stackrel{?}{=}$ coNP
- NP $\cap$ coNP $\stackrel{?}{=}$ P

Polynomials

In algebraic combinatorics and combinatorial representation theory we often study:

$$F_\diamond = \sum_\alpha c_{\alpha,\diamond} x^\alpha = \sum_{s \in S} \mathrm{wt}(s) \in \mathbb{Z}[x_1, \ldots, x_n]$$

Example 1: $\diamond = \lambda \implies F_\diamond = s_\lambda$ (Schur), $c_{\alpha,\lambda} = K_{\lambda,\alpha} =$ Kostka coeff.

Example 2: $\diamond = G = (V, E) \implies F_\diamond = \chi_G$ (Stanley's chromatic symmetric polynomial), $c_{\alpha,G} = \#$proper colorings of G with α_i-many colors i

Example 3: $\diamond = w \in S_\infty \implies F_\diamond = \mathfrak{S}_w$ (Schubert polynomial). More later.

The decision problem we care about: Nonvanishing

Nonvanishing: What is the complexity of deciding $c_{\alpha,\diamond} \neq 0$ as measured in the length of the input $(\alpha, \diamond)$ assuming arithmetic takes constant time?

- In general undecidable: Gödel incompleteness '31, Turing's halting problem '36.
- Our cases of interest have combinatorial positivity:
  $\exists$ rule for $c_{\alpha,\diamond} \in \mathbb{Z}_{\geq 0} \implies \text{Nonvanishing}(F_\diamond) \in \text{NP}$.

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Warning: Standard combinatorics might <u>not</u> be *manifestly* in NP.

Ex. Does this SSYT certify Kostka coeff. $K_{\lambda,\mu} \neq 0$ where $\lambda = (10^{100}, 10^{100})$ and $\mu = (0^{20}, 4, 3, 2, 1, 2, 1, 0^6, 2, \ldots)$?

21	21	21	22	25	25	25	27	28	36	36	36	37	37	⋯	53	53	53	54	54	55	56	56	57	⋯	71	72	75	75	79	91	91
22	22	22	23	26	26	28	28	29	37	37	37	39	39	⋯	54	54	54	55	55	56	57	57	58	⋯	72	73	76	76	80	92	97

This is a complexity rationale for Gelfand-Tsetlin polytopes.

Newton polytopes

Evidently, nonvanishing concerns the *Newton polytope*,

$$\mathrm{Newton}(F_\diamond) = \mathrm{conv}\{\alpha : c_{\alpha,\diamond} \neq 0\} \subseteq \mathbb{R}^n.$$

Definition: (Monical-Tokcan-Y.) $F_\diamond$ has *saturated Newton polytope* (S.N.P.) if $\beta \in \mathrm{Newton}(F_\diamond) \iff c_{\beta,\diamond} \neq 0$

- *Many* polynomials in algebraic comb. have this property.
- Application: A. Woo-Y. solves a complexity problem of D. Grigoriev-G. Koshevoy.
- Further work: subsets of {A. Fink, J. Huh, R. Liu, J. Matherne, K. Mészáros, A. St. Dizier}.
- Numerous open problems remain. For example:

Fact: (MTY) $\Delta_n := \prod_{1\leq i<j\leq n}(x_i - x_j)^2$ is S.N.P. $\iff n \leq 4$.

Conjecture: (MTY) Fix k, $\exists n$ such that Δ_n^k is not S.N.P.

Complexity and the S.N.P. property

Observation 1: S.N.P. $\Rightarrow$ nonvanishing$(F_\diamond)$ is equivalent to checking membership of a lattice point in Newton$(F_\diamond)$.

Observation 1': S.N.P. + "efficient" halfspace description of Newton$(F_\diamond)$ $\Longrightarrow$ nonvanishing$(F_\diamond) \in$ coNP.

$\therefore$ in many cases nonvanishing$(F_\diamond) \in$ NP $\cap$ coNP.

Nonvanishing and NP

Example 1': s_λ has S.N.P. Newton$(s_\lambda) = \mathcal{P}_\lambda$ (the permutahedron). Nonvanishing$(s_\lambda) \in$ P by dominance order (Rado's theorem).

Example 2': χ_G does not have S.N.P..

coloring $\in$ NP-complete $\implies$ Nonvanishing$(\chi_G) \in$ NP-complete.

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Example 2': χ_G does not have S.N.P..

$$\text{coloring} \in \text{NP-complete} \implies \text{Nonvanishing}(\chi_G) \in \text{NP-complete.}$$

$\therefore$ nonvanishing hits the extremes of NP.

Question: What about the nonextremes?

- Many problems *suspected* of being NP-intermediate: e.g., graph isomorphism, factorization
- Ladner's theorem: P $\neq$ NP $\implies$ NP-intermediate $\neq \emptyset$
- Problems in NP $\cap$ coNP are suspects for NP-intermediate since

$$\text{coNP} \cap \text{NP-complete} \neq \emptyset \implies \text{NP} = \text{coNP!}$$

- This is why factorization is not expected to be NP-complete.

Conjecture 1: [Stanley '95] If G is claw-free (i.e., it contains no induced $K_{1,3}$ subgraph), then χ_G is Schur positive.

Conjecture 2: [C. Monical '18] If χ_G is Schur positive, then it is SNP.

Conjecture 1+2: If G is claw-free then χ_G is SNP.

Theorem: (Holyer '81) Coloring of claw-free G is NP-complete.

Corollary: nonvanishing$(\chi_{\text{claw-free}\,G}) \in$ NP-complete.

Proposition: (Adve-Robichaux-Y. '18) Conjecture 1+2 and a halfspace description of Newton$(\chi_{\text{clawfree}\,G}) \implies$ NP $=$ coNP

Suggests a new complexity-theoretic rationale for the study of χ_G.

An algebraic combinatorics paradigm for complexity

In many cases of algebraic combinatorics, $\{F_\diamond\}$ has combinatorial positivity and SNP. If one also has an efficient halfspace description of $\mathrm{Newton}(F_\diamond)$, then $\mathrm{nonvanishing}(F_\diamond) \in \mathrm{NP} \cap \mathrm{coNP}$.

Three *plausible* outcomes of such a study:

(I) **Unknown**: it is an open problem to find additional problems that are in $\mathrm{NP} \cap \mathrm{coNP}$ that are not *known* to be in P.

(II) **P**: Give an algorithm. It will likely illuminate some special structure, of independent combinatorial interest.

(III) **NP-complete**: (conjecturally) implies $\mathrm{NP} \stackrel{?}{=} \mathrm{coNP}$ with "$=$".

Your favorite polynomial family to think about this way?

My favorite is Schubert polynomials. Initially Adve, Robichaux and I got to outcome (I), but then achieved outcome (II).

Schubert polynomials

B acts on GL_n/B with *finitely many orbits*, the Schubert cells, whose closures X_w, $w \in S_n$ are the **Schubert varieties**.

Lascoux and Schützenberger's (1982) main idea in type A (after Bernstein-Gelfand-Gelfand):

- Pick $\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ as an especially nice representative of the class of a point
- Apply *Newton's divided difference operator*

$$\partial_i f = \frac{f - f^{s_i}}{x_i - x_{i+1}},$$

to recursively define all other $\mathfrak{S}_w$ using weak Bruhat order.

This starts the theory of *Schubert polynomials*.

Complexity results

There are many combinatorial rules that establish that $c_{\alpha,w} \in \mathbb{Z}_{\geq 0}$.

However, none of these prove nonvanishing$(\mathfrak{S}_w) \in \mathsf{P}$ since they involve exponential search.

Theorem A: (Adve-Robichaux-Y. '18) $c_{\alpha,w}$ is $\#$P-complete.

$\therefore$ no poly. time algorithm to compute $c_{\alpha,w}$ exists unless $\mathsf{P} = \mathsf{NP}$.

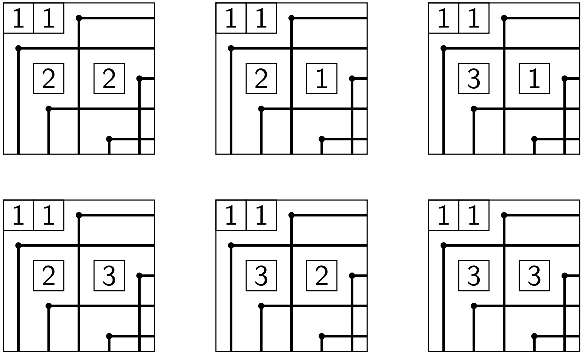
Counting is hard, nonvanishing is easy:

Theorem B: (Adve-Robichaux-Y. '18) nonvanishing$(\mathfrak{S}_w) \in \mathsf{P}$

Analogy: Computing the permanent of a $0,1$-matrix is $\#$P-complete but nonzeroness is easy (Edmonds-Karp matching algorithm).

A tableau rule for nonvanishing

Fillings of the Rothe diagram of 31524:



Theorem C: (Adve-Robichaux-Y. '18)
$c_{\alpha,w} \neq 0 \iff \mathrm{Tab}(w,\alpha) \neq \emptyset$.

Proofs

- The *Schubitope* $\mathcal{S}_D$ was introduced by Monical-Tokcan-Y. for any $D \subseteq [n]^2$.
- We give a generalization of tableau of Theorem C to any D.
- Then introduce a new polytope $\mathcal{T}_D$ whose integer points biject with tableaux.
- Integer linear programming is hard but $\mathcal{T}_D$ is totally unimodular. Now use LPfeasibility $\in$ P.
- Link to Schubert polynomials we use:

Conjecture (MTY) For $D = D(w)$, $\mathcal{S}_D = \text{Newton}(\mathfrak{S}_w)$ and $\mathfrak{S}_w$ is S.N.P.

Theorem (Fink-Mészáros-St. Dizier '18): The above conjecture is true.

- NP and $\#P$ proof via transition.

Conclusions and summary

- In this talk we described an *algebraic* combinatorics paradigm for complexity on theoretical computer science.
- Conversely, complexity gives some new perspectives on algebraic combinatorics (Stanley's chromatic symmetric polynomials).
- In our main example, we obtain new results about Schubert polynomials and the Schubitope.

More $F_\diamond$'s in algebraic combinatorics deserve analysis of $\mathrm{Newton}(F_\diamond)$ and $\mathrm{Nonvanishing}(F_\diamond)$.