Complexity, Combinatorial Positivity, and Newton Polytopes

Alexander Yong
University of Illinois at Urbana-Champaign

Based on joint work with:

Anshul Adve (University of California at Los Angeles)
Colleen Robichaux (University of Illinois at Urbana-Champaign)
- and -
C. Monical (Sandia National Labs)
N. Tokcan (Broad Institute of MIT and Harvard)
**Poorly understood issue:** Why are some decision problems have fast algorithms and others seem to need costly search?

Multiplication is easy:

\[
90912135295978188784406583026004374858926083103 \\
28358720428512168960411528640933367824950788367 \\
956756806141 \times \ 814385925911004526572780912628442 \\
93358778990021676278832009141724293243601330041 \\
16702003240828777970252499
\]
**Poorly understood issue:** Why are some decision problems have fast algorithms and others seem to need costly search?

Multiplication is easy:

\[
90912135295978188784406583026004374858926083103 \\
28358720428512168960411528640933367824950788367 \\
956756806141 \times 814385925911004526572780912628442 \\
93358778990021676278832009141724293243601330041 \\
16702003240828777970252499
\]

Factoring seems hard. RSA $30,000$ challenge:

\[
74037563479561712828046796097429573142593188889 \\
23128908493623263897276503402826627689199641962 \\
51178439958943305021275853701189680982867331732 \\
73108930900552505116877063299072396380786710086 \\
096962537934650563796359
\]

Solved in 2012.
Complexity has long connections of combinatorics, but mainly *graph theory* and *optimization*. We’d like to propose a paradigm for *algebraic* combinatorics to connect to complexity.
Computational Complexity Theory II

Complexity has long connections of combinatorics, but mainly *graph theory* and *optimization*. We’d like to propose a paradigm for *algebraic* combinatorics to connect to complexity.

∴ I now give a brief summary of complexity theory:

- **NP**: LP ($\exists x \geq 0, Ax=b$?)
- **coNP**: Primes
- **P**: LP and Primes!
- **NP-complete**: Graph coloring

Famous theoretical computer science problems relevant to us:

- $P \overset{?}{=} NP$
- $NP \overset{?}{=} coNP$
- $NP \cap coNP \overset{?}{=} P$
In algebraic combinatorics and combinatorial representation theory we often study:

\[ F_{\diamond} = \sum_{\alpha} c_{\alpha,\diamond} x^\alpha = \sum_{s \in S} \text{wt}(s) \in \mathbb{Z}[x_1, \ldots, x_n] \]

**Example 1:** \( \diamond = \lambda \implies F_{\diamond} = s_\lambda \) (Schur), \( c_{\alpha,\lambda} = K_{\lambda,\alpha} = \text{Kostka coeff.} \)

**Example 2:** \( \diamond = G = (V, E) \implies F_{\diamond} = \chi_G \) (Stanley’s chromatic symmetric polynomial), \( c_{\alpha,G} = \#\) proper colorings of \( G \) with \( \alpha_i \)-many colors \( i \)

**Example 3:** \( \diamond = w \in S_\infty \implies F_{\diamond} = \mathcal{G}_w \) (Schubert polynomial). More later.
The decision problem we care about: Nonvanishing

**Nonvanishing:** What is the complexity of deciding $c_{\alpha,\diamond} \neq 0$ as measured in the length of the input $(\alpha, \diamond)$ assuming arithmetic takes constant time?

- In general **undecidable**: Gödel incompleteness '31, Turing’s halting problem '36.

- Our cases of interest have combinatorial positivity:
  - $\exists$ rule for $c_{\alpha,\diamond} \in \mathbb{Z}_{\geq 0} \implies \text{Nonvanishing}(F_\diamond) \in \text{NP}$.
The decision problem we care about: Nonvanishing

**Nonvanishing**: What is the complexity of deciding $c_{\alpha, \diamond} \neq 0$ as measured in the length of the input $(\alpha, \diamond)$ assuming arithmetic takes constant time?

- In general **undecidable**: Gödel incompleteness ’31, Turing’s halting problem ’36.
- Our cases of interest have combinatorial positivity: 
  \[ \exists \text{ rule for } c_{\alpha, \diamond} \in \mathbb{Z}_{\geq 0} \implies \text{Nonvanishing}(F_{\diamond}) \in \text{NP}. \]

**Warning**: Standard combinatorics might **not** be *manifestly* in NP.

Ex. Does this SSYT certify Kostka coeff. $K_{\lambda, \mu} \neq 0$ where 
$\lambda = (10^{100}, 10^{100})$ and $\mu = (0^{20}, 4, 3, 2, 1, 2, 1, 0^6, 2, \ldots)$?

\[
\begin{align*}
21 & 121212225252527283636363737 \ldots \\
22 & 2222222326262828293737373939 \ldots
\end{align*}
\]

This is a complexity rationale for Gelfand-Tsetlin polytopes.
Evidently, nonvanishing concerns the Newton polytope,
\[
\text{Newton}(F_{\diamond}) = \text{conv}\{\alpha : c_{\alpha,\diamond} \neq 0\} \subseteq \mathbb{R}^n.
\]

**Definition:** (Monical-Tokcan-Y.) \( F_{\diamond} \) has saturated Newton polytope (S.N.P.) if \( \beta \in \text{Newton}(F_{\diamond}) \iff c_{\beta,\diamond} \neq 0 \)

- Many polynomials in algebraic comb. have this property.
- Further work: subsets of \{A. Fink, J. Huh, R. Liu, J. Matherne, K. Mészáros, A. St. Dizier\}.
- Numerous open problems remain. For example:

**Fact:** (MTY) \( \Delta_n := \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \) is S.N.P. \( \iff n \leq 4. \)

**Conjecture:** (MTY) Fix \( k, \exists n \) such that \( \Delta^k_n \) is not S.N.P.
**Observation 1:** S.N.P. \(\Rightarrow\) nonvanishing\((F_\diamond)\) is equivalent to checking membership of a lattice point in \(\text{Newton}(F_\diamond)\).

**Observation 1’:** S.N.P. + “efficient” halfspace description of \(\text{Newton}(F_\diamond)\) \(\implies\) nonvanishing\((F_\diamond)\) \(\in\) coNP.

\[
\therefore \text{in many cases nonvanishing}(F_\diamond) \in \text{NP} \cap \text{coNP}.
\]
Nonvanishing and NP

**Example 1’**: \( s_\lambda \) has S.N.P. \( \text{Newton}(s_\lambda) = P_\lambda \) (the permutahedron). Nonvanishing \( (s_\lambda) \in P \) by dominance order (Rado’s theorem).

**Example 2’**: \( \chi_G \) does not have S.N.P.

\[ \text{coloring} \in \text{NP-complete} \implies \text{Nonvanishing}(\chi_G) \in \text{NP-complete}. \]

\[ \therefore \text{nonvanishing hits the extremes of NP}. \]
Example 1’: $s_{\lambda}$ has S.N.P. Newton($s_{\lambda}$) = $P_{\lambda}$ (the permutahedron). Nonvanishing($s_{\lambda}$) $\in$ P by dominance order (Rado’s theorem).

Example 2’: $\chi_{G}$ does not have S.N.P..

coloring $\in$ NP-complete $\implies$ Nonvanishing($\chi_{G}$) $\in$ NP-complete.

∴ nonvanishing hits the extremes of NP.

Question: What about the nonextremes?

- Many problems suspected of being NP-intermediate: e.g., graph isomorphism, factorization
- Ladner’s theorem: $P \neq NP$ $\implies$ NP-intermediate $\neq \emptyset$
- Problems in $NP \cap coNP$ are suspects for NP-intermediate since

$$coNP \cap NP\text{-}complete \neq \emptyset \implies NP = coNP!$$

This is why factorization is not expected to be NP-complete.
Conjecture 1: [Stanley ’95] If $G$ is claw-free (i.e., it contains no induced $K_{1,3}$ subgraph), then $\chi_G$ is Schur positive.

Conjecture 2: [C. Monical ’18] If $\chi_G$ is Schur positive, then it is SNP.

Conjecture 1+2: If $G$ is claw-free then $\chi_G$ is SNP.

Theorem: (Holyer ’81) Coloring of claw-free $G$ is NP-complete.

Corollary: nonvanishing($\chi_{\text{claw-free}G}$) $\in$ NP-complete.

Proposition: (Adve-Robichaux-Y. ’18) Conjecture 1+2 and a halfspace description of $\text{Newton}(\chi_{\text{claw-free}G}) \implies \text{NP} = \text{coNP}$

Suggests a new complexity-theoretic rationale for the study of $\chi_G$. 
In many cases of algebraic combinatorics, \( \{F_\Diamond\} \) has combinatorial positivity and SNP. If one also has an efficient halfspace description of Newton\((F_\Diamond)\), then nonvanishing\((F_\Diamond) \in \text{NP} \cap \text{coNP}\).

Three plausible outcomes of such a study:

(I) **Unknown**: it is an open problem to find additional problems that are in \( \text{NP} \cap \text{coNP} \) that are not *known* to be in \( \text{P} \).

(II) **P**: Give an algorithm. It will likely illuminate some special structure, of independent combinatorial interest.

(III) **NP-complete**: (conjecturally) implies \( \text{NP} = \text{coNP} \) with “=”.

Your favorite polynomial family to think about this way?

My favorite is Schubert polynomials. Initially Adve, Robichaux and I got to outcome (I), but then achieved outcome (II).
$B$ acts on $GL_n/B$ with \textit{finitely many orbits}, the Schubert cells, whose closures $X_w, w \in S_n$ are the \textbf{Schubert varieties}.

Lascoux and Schützenberger’s (1982) main idea in \textit{type $A$} (after Bernstein-Gelfand-Gelfand):

- Pick $S_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ as an especially nice representative of the class of a point
- Apply \textit{Newton’s divided difference operator}

\[
\partial_i f = \frac{f - f^{s_i}}{x_i - x_{i+1}},
\]

\[x \in \mathbb{R}^n \setminus \{x_i = x_{i+1} \}.
\]

\hspace{1cm}
to recursively define all other $S_w$ using weak Bruhat order.

This starts the theory of \textbf{Schubert polynomials}. 

\hspace{1cm}
Complexity results

There are many combinatorial rules that establish that $c_{\alpha,w} \in \mathbb{Z}_{\geq 0}$.

However, none of these prove nonvanishing $(S_w) \in P$ since they involve exponential search.

**Theorem A:** (Adve-Robichaux-Y. ’18) $c_{\alpha,w}$ is $\#P$-complete.

∴ no poly. time algorithm to compute $c_{\alpha,w}$ exists unless $P = NP$.

Counting is hard, nonvanishing is easy:

**Theorem B:** (Adve-Robichaux-Y. ’18) nonvanishing $(S_w) \in P$

**Analogy:** Computing the permanent of a 0, 1-matrix is $\#P$-complete but nonzeroness is easy (Edmonds-Karp matching algorithm).
A tableau rule for nonvanishing Fillings of the Rothe diagram of 31524:

- Theorem C: (Adve-Robichaux-Y. ’18)
\[ c_{\alpha,w} \neq 0 \iff \text{Tab}(w, \alpha) \neq \emptyset. \]
The Schubitope $S_D$ was introduced by Monical-Tokcan-Y. for any $D \subseteq [n]^2$.

We give a generalization of tableau of Theorem C to any $D$.

Then introduce a new polytope $T_D$ whose integer points biject with tableaux.

Integer linear programming is hard but $T_D$ is totally unimodular. Now use LPfeasibility $\in P$.

Link to Schubert polynomials we use:

**Conjecture** (MTY) For $D = D(w)$, $S_D = \text{Newton}(\mathcal{G}_w)$ and $\mathcal{G}_w$ is S.N.P.

**Theorem** (Fink-Mészáros-St. Dizier ’18): The above conjecture is true.

NP and $\#P$ proof via transition.
In this talk we described an *algebraic* combinatorics paradigm for complexity on theoretical computer science.

Conversely, complexity gives some new perspectives on algebraic combinatorics (Stanley’s chromatic symmetric polynomials).

In our main example, we obtain new results about Schubert polynomials and the Schubitope.

More $F_\diamond$’s in algebraic combinatorics deserve analysis of $\text{Newton}(F_\diamond)$ and $\text{Nonvanishing}(F_\diamond)$.