Hoffman constants for systems of linear inequalities

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Triangle Lectures in Combinatorics
North Carolina State University
November 16, 2019

Main ideas

Error bounds: cases where "almost" implies "near"

Suppose $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

Linear equations $Ax = b$

If $A^{-1}(b) := \{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$ then for all $u \in \mathbb{R}^n$

$$\|Au - b\| \text{ small } \Rightarrow u \text{ near } A^{-1}(b)$$

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Linear inequalities (Hoffman's error bound) $Ax \leq b$

If $P_A(b) := \{x \in \mathbb{R}^n : Ax \leq b\} \neq \emptyset$ then for all $u \in \mathbb{R}^n$

$$\|(Au - b)_+\| \text{ small } \Rightarrow u \text{ near } P_A(b).$$

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Linear equations relative to a set $Ax = b, \; x \in R$

Suppose $R \subseteq \mathbb{R}^n$ and $b \in A(R)$, that is, $A^{-1}(b) \cap R \neq \emptyset$. Is it the case that for all $u \in R$

$$\|Au - b\| \text{ small } \Rightarrow u \text{ near } A^{-1}(b) \cap R?$$

Agenda

- Error bound for $Ax = b$
- Hoffman's error bound for $Ax \leq b$
- Error bounds for $Ax = b, \ x \in R$ and for $Ax \leq b, \ x \in R$
- Condition number relative to a reference set
- Algorithms to compute Hoffman constants

Error bound for systems of linear equations

Systems of linear equations $Ax = b$

Suppose $\mathbb{R}^n$ and $\mathbb{R}^m$ be endowed with norms and $A \in \mathbb{R}^{m\times n} \setminus \{0\}$.

Then for all $b \in A(\mathbb{R}^n) := \{Ax : x \in \mathbb{R}^n\}$ and all $u \in \mathbb{R}^n$

$$\operatorname{dist}(u, A^{-1}(b)) \leq \|A^{-1}\| \cdot \|Au - b\| \tag{1}$$

where

$$\|A^{-1}\| = \max_{\substack{y\in A(\mathbb{R}^n)\\ \|y\|\leq 1}} \min_{\substack{x\in\mathbb{R}^n\\ Ax=y}} \|x\| = \frac{1}{\displaystyle\min_{v\in A(\mathbb{R}^n),\, \|v\|^*=1} \|A^{\mathsf{T}} v\|^*}.$$

Systems of linear equations $Ax = b$

Suppose $\mathbb{R}^n$ and $\mathbb{R}^m$ be endowed with norms and $A \in \mathbb{R}^{m \times n} \setminus \{0\}$.

Then for all $b \in A(\mathbb{R}^n) := \{Ax : x \in \mathbb{R}^n\}$ and all $u \in \mathbb{R}^n$

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Inequality (1) is an *error bound* for $Ax = b$

If u "almost" solves $Ax = b$ then u is "near" a solution to $Ax = b$.

Inequality (1) is tight: there exist $b \in A(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$

$$\operatorname{dist}(u, A^{-1}(b)) = \|A^{-1}\| \cdot \|Au - b\| > 0.$$

Recall: $\|v\|^* = \max_{\|y\| \le 1} v^\mathsf{T} y$. For instace, $\|\cdot\|_1^* = \|\cdot\|_\infty$ and $\|\cdot\|_2^* = \|\cdot\|_2$

Geometric interpretation of $\|A^{-1}\|$

Suppose $A \in \mathbb{R}^{m\times n} \setminus \{0\}$. Then

$$\frac{1}{\|A^{-1}\|} = \mathsf{dist}\,(0, \mathsf{relbdy}(\{Ax : \|x\| \le 1\}))$$

and

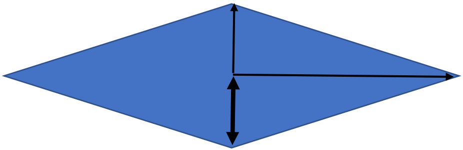
$$\frac{1}{\|A^{-1}\|} = \mathsf{dist}^*\left(0, \{A^\mathsf{T} v : v \in A(\mathbb{R}^n), \|v\|^* = 1\}\right).$$

For ℓ_2 norms $\frac{1}{\|A^{-1}\|}$ = smallest positive singular value of A.

Example

Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$ for $0 \le \epsilon < 1$ and $\mathbb{R}^2$ is endowed with the ℓ_1 norm. Then

$$\frac{1}{\|A^{-1}\|} = \begin{cases} \epsilon & \text{if } \epsilon \in (0,1) \\ 1 & \text{if } \epsilon = 0. \end{cases}$$



$\{Ax : \|x\| \le 1\}$ and $1/\|A^{-1}\|$ for $0 < \epsilon < 1$.

$\{Ax : \|x\| \le 1\}$ and $1/\|A^{-1}\|$ for $\epsilon = 0$.

Hoffman's error bound for systems of linear inequalities

Systems of linear inequalities $Ax \leq b$

Key notation

For $A \in \mathbb{R}^{m\times n},\ b \in A(\mathbb{R}^n) + \mathbb{R}^m_+$, and $u \in \mathbb{R}^n$

$$P_A(b) := \{x \in \mathbb{R}^n : Ax \leq b\}.$$

Theorem (Hoffman 1952)

Let $A \in \mathbb{R}^{m\times n}$. Then there exists a constant $H(A)$ that depends only on A such that for all $b \in A(\mathbb{R}^n) + \mathbb{R}^m_+$ and all $u \in \mathbb{R}^n$

$$\mathrm{dist}(u, P_A(b)) \leq H(A) \cdot \mathrm{dist}(b - Au, \mathbb{R}^m_+).$$

For many norms: $\mathrm{dist}(b - Au, \mathbb{R}^m_+) = \|(b - Au)_-\| = \|(Au - b)_+\|$.

Characterization of $H(A)$ (special case)

Suppose $A \in \mathbb{R}^{m\times n}$ is such that $A(\mathbb{R}^n) + \mathbb{R}^m_+ = \mathbb{R}^m$. Let

$$H(A) := \max_{\substack{y\in\mathbb{R}^m \\ \|y\|\le 1}} \min_{\substack{x\in\mathbb{R}^n \\ Ax\le y}} \|x\| = \frac{1}{\displaystyle\min_{v\in\mathbb{R}^m_+,\, \|v\|^*=1} \|A^\mathsf{T} v\|^*}.$$

Characterization of $H(A)$ (special case)

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Proposition (Following Renegar 1995, Ramdas & P 2015)

Suppose $A \in \mathbb{R}^{m \times n}$ is such that $A(\mathbb{R}^n) + \mathbb{R}^m_+ = \mathbb{R}^m$. Then for all $b \in \mathbb{R}^m$ and all $u \in \mathbb{R}^n$

$$\mathsf{dist}(u, P_A(b)) \le H(A) \cdot \mathsf{dist}(b - Au, \mathbb{R}^m_+).$$

This bound is tight: there exist $b \in \mathbb{R}^m$ and $u \in \mathbb{R}^n$ such that

$$\mathsf{dist}(u, P_A(b)) = H(A) \cdot \mathsf{dist}(b - Au, \mathbb{R}^m_+) > 0.$$

Observe: $A(\mathbb{R}^n) + \mathbb{R}^m_+ = \mathbb{R}^m \Leftrightarrow Ax < 0$ is feasible.

Geometric interpretation of $H(A)$ in this special case

Observe

$$
\begin{aligned}
A(\mathbb{R}^n) + \mathbb{R}^m_+ = \mathbb{R}^m &\Leftrightarrow Ax < 0 \text{ is feasible} \\
&\Leftrightarrow A^\mathsf{T} v = 0,\ v \gneq 0 \text{ infeasible.}
\end{aligned}
$$

Second step by Gordan's Theorem.

Therefore

$$A(\mathbb{R}^n) + \mathbb{R}^m_+ = \mathbb{R}^m \Leftrightarrow 0 \notin \{A^\mathsf{T} v : v \geq 0, \|v\|^* = 1\}.$$

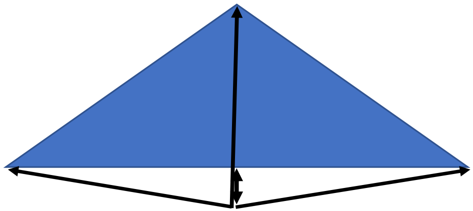
When this is the case we have

$$\frac{1}{H(A)} = \mathsf{dist}^* \left(0, \{A^\mathsf{T} v : v \geq 0, \|v\|^* = 1\}\right).$$

Example

Suppose $A = \begin{bmatrix} -1 & \epsilon \\ 1 & \epsilon \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$ for $0 < \epsilon < 1$ and both $\mathbb{R}^3$ and $\mathbb{R}^2$ are endowed with the ℓ_∞ norm. Then

$$\frac{1}{H(A)} = \epsilon.$$



$\{A^\mathsf{T} v : v \geq 0,\ \|v\|^* = 1\}$ and $1/H(A)$ for $\epsilon \in (0, 1)$.

Characterization of $H(A)$ (general case)

Let $\mathcal{J}(A) := \{J \subseteq [m] : A_J(\mathbb{R}^n) + \mathbb{R}^J_+ = \mathbb{R}^J\}$ and

$$H(A) := \max_{J \in \mathcal{J}(A)} H(A_J)$$
$$= \max_{J \in \mathcal{J}(A)} \frac{1}{\min\{\|A_J^{\mathsf{T}} v\|^* : v \in \mathbb{R}^J_+, \|v\|^* = 1\}}.$$

(Here $[m]$ is shorthand for $\{1, \ldots, m\}$.)

Characterization of $H(A)$ (general case)

Let $\mathcal{J}(A) := \{J \subseteq [m] : A_J(\mathbb{R}^n) + \mathbb{R}^J_+ = \mathbb{R}^J\}$ and

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(Here $[m]$ is shorthand for $\{1, \ldots, m\}$.)

Proposition

Let $A \in \mathbb{R}^{m \times n} \setminus \{0\}$. Then for all $b \in A(\mathbb{R}^n) + \mathbb{R}^m_+$ and all $u \in \mathbb{R}^n$

$$\mathsf{dist}(u, P_A(b)) \leq H(A) \cdot \mathsf{dist}(b - Au, \mathbb{R}^m_+).$$

Furthermore, this bound is tight.

Related results by Robinson 1973, Li 1993, Klatte & Thiere 1995, Wang & Lin 2014.

Error bounds relative to a reference set

Relative error bound for equations

Consider a system of linear constraints of the form

$$\begin{aligned} Ax &= b \\ x &\in R \end{aligned}$$

where R is a "reference" set representing some easy-to-satisfy constraints, e.g., $x \geq 0$, box constraints, etc.

It is natural to consider an error bound that accounts for this kind of reference set.

Goal

For $u \in R$, bound $\text{dist}(u, A^{-1}(b) \cap R)$ in terms of $\|Au - b\|$.

Why consider relative error bounds?

Because they can be drastically different from regular error bounds.

Example

Suppose $A = \begin{bmatrix} I_{n-1} & 0 \\ 0 & \epsilon \end{bmatrix}$ for $0 < \epsilon \ll 1$ and we work with ℓ_1 norms. For $b \in A(\mathbb{R}^n)$ we have

$$\mathrm{dist}(u, A^{-1}(b)) \leq \frac{1}{\epsilon} \cdot \|Au - b\|.$$

Now suppose $R = \mathbb{R}^{n-1} \times \{0\}$. For $b \in A(R)$ and $u \in R$

$$\mathrm{dist}(u, A^{-1}(b) \cap R) \leq \|Au - b\|.$$

Why consider relative error bounds?

Because they can be drastically different from regular error bounds.

Example

Suppose $A = \begin{bmatrix} 1 & -1 & 0 \\ -\epsilon & -\epsilon & 1 \end{bmatrix}$ for $0 < \epsilon \ll 1$ and we work with ℓ_1 norms. For $b \in A(\mathbb{R}^3)$ we have

$$\mathsf{dist}(u, A^{-1}(b)) \leq \|Au - b\|.$$

Now suppose $R = \mathbb{R}^3_+$. For $b \in A(R)$ and $u \in R$

$$\mathsf{dist}(u, A^{-1}(b) \cap R) \leq \frac{1}{\epsilon} \cdot \|Au - b\|.$$

Relative error bound for equations (special case)

Suppose $A \in \mathbb{R}^{m\times n}$ and $R \subseteq \mathbb{R}^n$ is a closed convex cone such that $A(R) := \{Ax : x \in R\}$ is a linear subspace. Let

$$H(A|R) := \max_{\substack{y\in A(R)\\ \|y\|\le 1}} \min_{\substack{x\in R\\ Ax=y}} \|x\| = \frac{1}{\displaystyle\min_{\substack{v\in A(R),\, \|v\|^*=1\\ A^{\mathsf{T}}v-u\in R^*}} \|u\|^*}.$$

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Proposition (Following Renegar 1995, Ramdas & P 2015)

Let $A \in \mathbb{R}^{m\times n}$ and $R \subseteq \mathbb{R}^n$ be a closed convex cone such that $A(R)$ is a linear subspace. Then for all $b \in A(R)$ and all $u \in R$

$$\mathrm{dist}(u, A^{-1}(b) \cap R) \le H(A|R) \cdot \|Au - b\|.$$

Furthermore, this bound is tight.

Geometric interpretation of $H(A|R)$ in this special case

Observe

Suppose $A \in \mathbb{R}^{m \times n}$ and $R \subseteq \mathbb{R}^n$ is a closed convex cone. Then

$$
\begin{aligned}
A(R) \text{ is a linear subspace} &\Leftrightarrow 0 \in \operatorname{relint}\left(\{Ax : x \in R, \|x\| \leq 1\}\right) \\
&\Leftrightarrow Ax = 0, x \in \operatorname{relint}(R) \text{ is feasible.}
\end{aligned}
$$

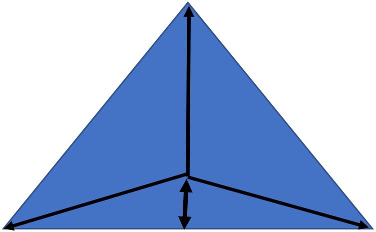
When this is the case we have

$$
\frac{1}{H(A|R)} = \operatorname{dist}\left(0, \operatorname{relbdy}(\{Ax : x \in R, \|x\| \leq 1\})\right).
$$

Example

Suppose $A = \begin{bmatrix} -1 & 1 & 0 \\ -\epsilon & -\epsilon & 1 \end{bmatrix} \in \mathbb{R}^{2\times 3}$ for $0 < \epsilon < 1/2$, both $\mathbb{R}^3$ and $\mathbb{R}^2$ are endowed with the ℓ_1 norm, and $R = \mathbb{R}^3_+$. Then

$$\frac{1}{H(A|R)} = \epsilon.$$



$\{Ax : \|x\| \leq 1\}$ and $1/H(A|R)$ for $\epsilon \in (0, 1)$.

Relative error bound for equations when R is polyhedral

Sets $\mathcal{T}(R)$ and $\mathcal{T}(A|R)$ of tangent cones

Suppose R is a polyhedron. For $u \in \mathbb{R}$, let $T_R(u)$ denote the *tangent cone* to R at $u \in R$, that is,

$$T_R(u) = \{d \in \mathbb{R}^d : u + td \in R \text{ for some } t > 0\}.$$

Let $\mathcal{T}(R) := \{T_R(u) : u \in R\}$ and

$$\mathcal{T}(A|R) := \{K \in \mathcal{T}(R) : A(K) \text{ is a linear subspace}\}.$$

Example

If $R = \mathbb{R}^n_+$ then $K \in \mathcal{T}(R)$ iff there exists $I \subseteq [n]$ such that

$$K = \{x \in \mathbb{R}^n : x_I \geq 0\}.$$

In this case $K \in \mathcal{T}(A|R)$ iff $Ax = 0, x_I > 0$ is feasible.

Relative error bound for equations when R is polyhedral

Suppose $A \in \mathbb{R}^{m \times n}$ and $R \subseteq \mathbb{R}^n$ is a reference polyhedron. Let

$$H(A|R) := \max_{K \in \mathcal{T}(A|R)} H(A|K).$$

Theorem (P, Vera, Zuluaga 2019)

Let $A \in \mathbb{R}^{m \times n}$ and $R \subseteq \mathbb{R}^n$ be a reference polyhedron. Then for all $b \in A(R)$ and all $u \in R$

$$\mathrm{dist}(u, A^{-1}(b) \cap R) \leq H(A|R) \cdot \|Au - b\|.$$

Furthermore, this bound is tight.

Relative error bound for inequalities

Consider a system of linear constraints of the form

$$\begin{aligned} Ax &\leq b \\ x &\in R \end{aligned}$$

where R is a reference set.

Recall

$P_A(b) = \{x \in \mathbb{R}^n : Ax \leq b\}$ so $P_A(b) \cap R = \{x \in R : Ax \leq b\}$.

Goal

For $u \in R$, bound $\text{dist}(u, P_A(b) \cap R)$ in terms of $\text{dist}(b - Au, \mathbb{R}^m_+)$.

Relative error bound for inequalities (special case)

Suppose $R \subseteq \mathbb{R}^n$ is a closed convex cone and $A \in \mathbb{R}^{m\times n}$ is such that $A(R) + \mathbb{R}^m_+ = \mathbb{R}^m$. Let

$$\tilde{H}(A|R) := \max_{\substack{y\in\mathbb{R}^m \\ \|y\|\le 1}} \min_{\substack{x\in R \\ Ax\le y}} \|x\| = \frac{1}{\min\limits_{\substack{v\in\mathbb{R}^m_+,\ \|v\|^*=1 \\ A^\mathsf{T} v - u \in R^*}} \|u\|^*}.$$

Proposition

Suppose $R \subseteq \mathbb{R}^n$ is a closed convex cone and $A \in \mathbb{R}^{m\times n}$ is such that $A(R) + \mathbb{R}^m_+ = \mathbb{R}^m$. Then for all $b \in \mathbb{R}^m$ and all $u \in R$

$$\mathsf{dist}(u, P_A(b) \cap R) \le \tilde{H}(A|R) \cdot \mathsf{dist}(b - Au, \mathbb{R}^m_+).$$

Furthermore this bound is tight.

Relative error bound for inequalities, R polyhedral

Suppose $A \in \mathbb{R}^{m\times n}$ and $R \subseteq \mathbb{R}^n$ is a reference polyhedron.

Recall that $[m] = \{1,\dots,m\}$ and $\mathcal{T}(R) = \{T_R(u) : u \in R\}$.

Let

$$\mathcal{J}(A|R) := \{(J,K) \in [m] \times \mathcal{T}(R) : A_J(K) + \mathbb{R}^J_+ = \mathbb{R}^J\}$$

and

$$\tilde{H}(A|R) := \max_{(J,K)\in\mathcal{J}(A|R)} \tilde{H}(A_J|K).$$

Theorem (P, Vera, Zuluaga 2019)

Let $A \in \mathbb{R}^{m\times n}$ and $R \subseteq \mathbb{R}^n$ be a reference polyhedron. Then for all $b \in A(R) + \mathbb{R}^m_+$ and all $u \in R$

$$\mathrm{dist}(u, P_A(b) \cap R) \le \tilde{H}(A|R) \cdot \mathrm{dist}(b - Au, \mathbb{R}^m_+).$$

Furthermore, this bound is tight.

Condition number relative to a reference set

Linear convergence of first-order algorithms

Consider the optimization problem

$$f^\star := \min_{x\in\mathbb{R}^n} \; f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable.

Gradient descent algorithm

$$x_{k+1} = x_k - t_k \nabla f(x_k) \text{ for some } t_k > 0$$

For illustration purposes, concentrate on the least-squares function

$$f(x) = \frac{1}{2}\|Ax - b\|_2^2$$

where $A \in \mathbb{R}^{m\times n},\; b \in \mathbb{R}^m$.

Linear convergence of first-order algorithms

Suppose $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m$ and $f(x) = \frac{1}{2}\|Ax - b\|_2^2$. Let $f^\star := \min_{x\in\mathbb{R}^n}\ f(x)$ and $X^\star := \operatorname{argmin}_{x\in\mathbb{R}^n}\ f(x)$.

Theorem

If $t_k > 0,\ k = 0, 1, \ldots$ are judiciously chosen (e.g., via exact line-search) then the gradient descent iterates satisfy

$$\mathsf{dist}(x_k, X^\star)^2 \leq \left(1 - \frac{1}{\mathsf{Cond}(f)}\right)^k \mathsf{dist}(x_0, X^\star)^2$$

and

$$f(x_k) - f^\star \leq \left(1 - \frac{1}{\mathsf{Cond}(f)}\right)^k (f(x_0) - f^\star),$$

where

$$\mathsf{Cond}(f) = \left(\|A\| \cdot \|A^{-1}\|\right)^2.$$

Above statement holds for f convex, differentiable and suitable $\mathsf{Cond}(f)$.

Linear convergence of first-order algorithms

Suppose $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ f(x) = \frac{1}{2}\|Ax-b\|_2^2$, and $R \subseteq \mathbb{R}^n$ is a reference polyhedron.

Consider the optimization problem

$$f^\star := \min_{x\in R}\ f(x).$$

Projected gradient descent algorithm

$$x_{k+1} = \Pi_R(x_k - t_k\nabla f(x_k)).$$

Here $\Pi_R : \mathbb{R}^n \to R$ denotes the orthogonal projection on R, i.e.,

$$\Pi_R(z) = \min_{x\in R}\|z-x\|_2.$$

Linear convergence of first-order algorithms

Suppose $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ f(x) = \frac{1}{2}\|Ax-b\|_2^2$, and $R \subseteq \mathbb{R}^n$ is a polyhedron. Let $f^\star := \min_{x\in R}\ f(x),\ X^\star :=$ argmin$_{x\in R}\ f(x)$.

Proposition (Gutman-P 2019 following Necoara et al 2018)

If $t_k > 0,\ k = 0,1,\ldots$ are judiciously chosen then the projected gradient descent iterates satisfy

$$\mathsf{dist}(x_k, X^\star)^2 \le \left(1 - \frac{1}{\mathsf{Cond}(f|R)}\right)^k \mathsf{dist}(x_0, X^\star)^2$$

and

$$f(x_k) - f^\star \le \left(1 - \frac{1}{\mathsf{Cond}(f|R)}\right)^k (f(x_0) - f^\star),$$

where

$$\mathsf{Cond}(f|R) = \left(\|A|\mathsf{span}(R-R)\| \cdot H(A|R)\right)^2.$$

Gutman-P 2019: Above holds for f convex, diff and suitable $\mathsf{Cond}(f|R)$.

Cond(f) versus Cond$(f|R)$

Suppose $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m$, and $f(x) = \frac{1}{2}\|Ax-b\|_2^2$. Recall

$$\mathsf{Cond}(f) = \left(\|A\|\cdot\|A^{-1}\|\right)^2.$$

If $R \subseteq \mathbb{R}^n$ is a reference polyhedron then

$$\mathsf{Cond}(f|R) = \left(\max_{K\in\mathcal{T}(A|R)} \|(A|K)\| \cdot \max_{K\in\mathcal{T}(A|R)} \|(A|K)^{-1}\|\right)^2,$$

where for each $K \in \mathcal{T}(A|R)$

$$\|(A|K)\| = \max_{\substack{x\in K\\ \|x\|\le 1}} \|Ax\| \ \text{ and } \ \|(A|K)^{-1}\| = \max_{\substack{y\in A(K)\\ \|y\|=1}} \min_{\substack{x\in K\\ y=Ax}} \|x\|.$$

The above expression for Cond$(f|R)$ holds because

$$\begin{aligned}\|A|\mathsf{span}(R-R)\| &= \max_{K\in\mathcal{T}(A|R)} \|(A|K)\|\\ H(A|R) &= \max_{K\in\mathcal{T}(A|R)} \|(A|K)^{-1}\|.\end{aligned}$$

Algorithms to compute $H(A)$

Computation of $H(A)$

Recall: for $A \in \mathbb{R}^{m \times n} \setminus \{0\}$

$$H(A) = \max_{J \in \mathcal{J}(A)} \frac{1}{\min\{\|A_J^\mathsf{T} v\|^* : v \in \mathbb{R}^J_+, \|v\|^* = 1\}},$$

where $\mathcal{J}(A) = \{J \subseteq [m] : A_J(\mathbb{R}^n) + \mathbb{R}^J_+ = \mathbb{R}^J\}$.

For judiciously chosen norms, the above expression for $H(A)$ can be formulated as a mixed integer linear program.

Can we do better?

Computation of $H(A)$ via the covering property

Let $\mathcal{F} \subseteq \mathcal{J}(A)$ and $\mathcal{I} \subseteq 2^{[m]} \setminus \mathcal{J}(A)$. Say that $(\mathcal{F}, \mathcal{I})$ satisfies the *covering property* if

> *For all $J \subseteq [m]$ either $J \subseteq F$ for some $F \in \mathcal{F}$ or $I \subseteq J$ for some $I \in \mathcal{I}$.*

Observation

If $(\mathcal{F}, \mathcal{I})$ satisfies the covering property then

$$H(A) = \max_{J \in \mathcal{F}} \frac{1}{\min\{\|A_J^\mathsf{T} v\|^* : v \in \mathbb{R}_+^J, \|v\|^* = 1\}}.$$

Idea for an algorithm

Gradually build $\mathcal{F} \subseteq \mathcal{J}(A)$ and $\mathcal{I} \subseteq 2^{[m]} \setminus \mathcal{J}(A)$ until $(\mathcal{F}, \mathcal{I})$ satisfies the covering property. Compute $H(A)$ via above formula.

Computation of $H(A)$ via the *covering property*

Key step: suppose J is not covered by $(\mathcal{F},\mathcal{I})$ and

$$v := \text{argmin}\{\|A_J^{\mathsf{T}}v\|^* : v \in \mathbb{R}_+^J, \|v\|^* = 1\}.$$

Observe that $\|A_J^{\mathsf{T}}v\|^* > 0 \Leftrightarrow J \in \mathcal{J}(A)$. Let $I(v) := \{i \in J : v_i > 0\}$.

Algorithm 1 Computation of $(\mathcal{F},\mathcal{I})$ and $H(A)$

Let $\mathcal{F} := \emptyset,\ \mathcal{I} := \emptyset, H(A) := 0$
while $(\mathcal{F},\mathcal{I})$ does not satisfy the covering property **do**
    Pick $J \in 2^{[m]}$ maximal not covered by $(\mathcal{F},\mathcal{I})$
    Let $v := \text{argmin}\{\|A_J^{\mathsf{T}}v\|^* : v \in \mathbb{R}_+^J, \|v\|^* = 1\}$.
    if $\|A_J^{\mathsf{T}}v\|^* > 0$ **then**
        $\mathcal{F} := \mathcal{F} \cup \{J\}$ and $H(A) := \max\left\{H(A), \frac{1}{\|A_J^{\mathsf{T}}v\|^*}\right\}$
    else
        Let $\mathcal{I} := \mathcal{I} \cup \{I(v)\}$
    end if
end while

Proposition

Let $A \in \mathbb{R}^{m\times n} \setminus \{0\}$ *and let* $\overline{\mathcal{J}}(A) \subseteq \mathcal{J}(A)$ *denote the maximal sets in* $\mathcal{J}(A)$ *and let* $\underline{\mathcal{J}}(A) \subseteq 2^{[m]} \setminus \mathcal{J}(A)$ *denote the minimal sets in* $2^{[m]} \setminus \mathcal{J}(A)$*. Algorithm 1 terminates after*

$$|\overline{\mathcal{J}}(A)| + |\underline{\mathcal{J}}(A)|$$

iterations. Upon termination Algorithm 1 returns $\mathcal{F} = \overline{\mathcal{J}}(A)$ *and* $\mathcal{I} = \underline{\mathcal{J}}(A)$*.*

Observe

Algorithm 1 terminates quickly if $\overline{\mathcal{J}}(A)$ has few and large sets.
Most favorable case: $\overline{\mathcal{J}}(A) = \{[m]\} \rightsquigarrow$ one iteration.
Next most favorable case: $\underline{\mathcal{J}}(A) = \{[m]\} \rightsquigarrow m+1$ iterations.

Examples

For convenience, suppose $\mathbb{R}^n$ and $\mathbb{R}^m$ are endowed with the ℓ_1-norm and ℓ_∞-norm respectively.

Example 1 (box)

For $A = \begin{bmatrix} \mathsf{I}_n \\ -\mathsf{I}_n \end{bmatrix} \in \mathbb{R}^{2n\times n}$ we have $H(A) = n$.

Example 2 (simplex)

For $A = \begin{bmatrix} \mathbf{1}_n^\mathsf{T} \\ -\mathsf{I}_n \end{bmatrix} \in \mathbb{R}^{(n+1)\times n}$ we have $H(A) = 2n + 1$.

Example 3 (ℓ_1-unit ball)

Let $A \in \mathbb{R}^{2^n\times n}$ be the matrix whose rows are the vectors with entries in $\{1, -1\}$. We computed the following values of $H(A)$:

n	1	2	3	4	5	6	7
$H(A)$	1	1	3	5	9	17	33

$\rightsquigarrow H(A) = 2^{n-2} + 1$ for $n \geq 3$?

Conclusions

- Error bounds ("almost" implies "near") for
  - linear equations $Ax = b$
  - linear inequalities $Ax \leq b$
  - linear equations/inequalities relative to a reference set
- Error bound constant in all cases is something like $\|A^{-1}\|$.
- Condition number of a function relative to a set.
- Algorithms to compute error bound constants.

Main references

- Peña, Vera, and Zuluaga (2019), New characterizations of Hoffman constants for systems of linear constraints, *ArXiv.*
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- Necoara, Nesterov, and Glineur (2018), Linear convergence of first-order methods for non-strongly convex optimization, *Math Prog.*
- Hoffman (1952), On approximate solutions of systems of linear inequalities, *J. of Research of National Bureau of Standards.*