Hoffman constants for systems of linear inequalities

Suppose $A \in \mathbb{R}^{m\times n}$.

Some notation

For $b \in \mathbb{R}^m$ let
$$A^{-1}(b) := \{x \in \mathbb{R}^n : Ax = b\}$$
and
$$P_A(b) := \{x \in \mathbb{R}^n : Ax \le b\}.$$

Main ideas

Suppose $u \in \mathbb{R}^n$. Then

- If $A^{-1}(b) \neq \emptyset$ and $\|Au - b\|$ is small then u is near $A^{-1}(b)$.
- If $P_A(b) \neq \emptyset$ and $\|(Au - b)_+\|$ is small then u is near $P_A(b)$.
- For "suitable" reference sets $R \subseteq \mathbb{R}^n$:
  If $A^{-1}(b) \cap R \neq \emptyset$, $u \in R$, and $\|Au - b\|$ is small then u is near $A^{-1}(b) \cap R$.

More notation

Let $\mathcal{J}(A)$ be the following collection of subsets of $[m] := \{1, \ldots, m\}$
$$\mathcal{J}(A) := \{J \subseteq [m] : A_J(\mathbb{R}^n) + \mathbb{R}^J_+ = \mathbb{R}^J\}$$
where $A_J \in \mathbb{R}^{J\times n}$ is the $|J| \times n$ submatrix of A defined by the rows in J.
For a polyhedron $R \subseteq \mathbb{R}^n$ and $u \in R$ let
$$T_R(u) := \{d \in \mathbb{R}^n : u + td \in R \text{ for some } t > 0\}.$$
Let $\mathcal{T}(R) := \{T_R(u) : u \in R\}$ and
$$\mathcal{T}(A|R) := \{K \in \mathcal{T}(R) : A(K) \text{ is a linear subspace}\}$$

References

- Peña, Vera, and Zuluaga (2019), New characterizations of Hoffman constants for systems of linear constraints, *ArXiv.*
- Gutman and Peña (2019), The condition number of a function relative to a set, *ArXiv.*
- Necoara, Nesterov, and Glineur (2018), Linear convergence of first-order methods for non-strongly convex optimization, *Math Prog.*
- Hoffman (1952), On approximate solutions of systems of linear inequalities, *J. of Research of National Bureau of Standards.*