Lattice congruences of the weak order: Algebra, combinatorics, and geometry

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Triangle Lectures in Combinatorics
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Winston-Salem, NC
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Main points

- The theory of lattice congruences of finite lattices (and particularly the weak order on a Coxeter group) is very combinatorial, and not very forbidding, and can uncover hidden combinatorial meaning/structures.
- Lattice congruences of the weak order have a nice discrete-geometric structure and meaning.
- There are connections to
  - cluster algebras,
  - representation theory of finite-dimensional algebras, and
  - mirror symmetry in algebraic geometry/string theory.

Section 1:   Lattice congruences for combinatorialists

Lattices

(Universal) algebra

A lattice is a set L with two binary operations $\wedge$ ("meet") and $\vee$ ("join") satisfying the axioms:

- $x \vee y = y \vee x$
- $x \wedge y = y \wedge x$
- $x \vee (y \vee z) = (x \vee y) \vee z$
- $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
- $x \vee (x \wedge y) = x$
- $x \wedge (x \vee y) = x$

Combinatorics

A lattice is a set L with a partial order "$\leq$" such that:

For all finite $S \subseteq L$,

- There exists a unique minimal upper bound for S is L, written $\bigvee S$.
- There exists a unique maximal lower bound for S is L, written $\bigwedge S$.

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$x \leq y$ iff $x \vee y = y$ iff $x \wedge y = x$

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$$x \vee y = \bigvee \{x, y\}$$

$$x \wedge y = \bigwedge \{x, y\}$$

Homomorphisms, congruences, quotients

(Lattice) homomorphism: a map $\eta : L_1 \to L_2$ such that

$$\eta(x \wedge y) = \eta(x) \wedge \eta(y) \text{ and } \eta(x \vee y) = \eta(x) \vee \eta(y).$$

Congruence: an equivalence relation $\equiv$ on L such that

$$(x_1 \equiv x_2 \text{ and } y_1 \equiv y_2) \implies (x_1 \wedge y_1 \equiv x_2 \wedge y_2 \text{ and } x_1 \vee y_1 \equiv x_2 \vee y_2).$$

Quotient: The set $L/\equiv$ of congruence classes with meet and join

$$[x] \vee [y] = [x \vee y] \text{ and } [x] \wedge [y] = [x \wedge y].$$

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$$[x] \vee [y] = [x \vee y] \text{ and } [x] \wedge [y] = [x \wedge y].$$

What do these mean in the order-theoretic definition of lattices?

Order-theoretic characterization of a lattice congruence

An equivalence relation $\equiv$ on a finite lattice L is a lattice congruence if and only if the following three conditions hold:

(i) Each equivalence class is an interval in L.

(ii) The map $\pi_\downarrow$ taking each element to the bottom element of its equivalence class is order-preserving.

(iii) The map $\pi^\uparrow$ taking each element to the top element of its equivalence class is order-preserving.

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A take-home lesson:

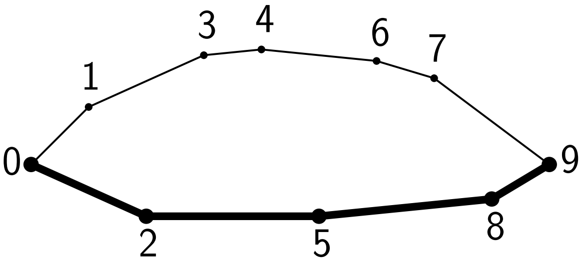
If you encounter a surjective set map $\eta : L \to S$ (a set):

- Check if the fibers (preimages of el'ts of S) are intervals in L.
- If so, check (ii) and (iii) on the fibers.
- If these hold, then the fibers of η are a congruence $\equiv$, and η induces a lattice structure on S, isomorphic to $L/\equiv$.

Example: Permutations-to-triangulations map

Arrange the numbers from 0 to $n + 1$ on a polygon such that numbers strictly increase left to right. Begin with a path along the bottom. Modify the path by removing/adding vertices in the order given by π. The triangulation is the union of the paths.

Example. $\pi = 42783165$



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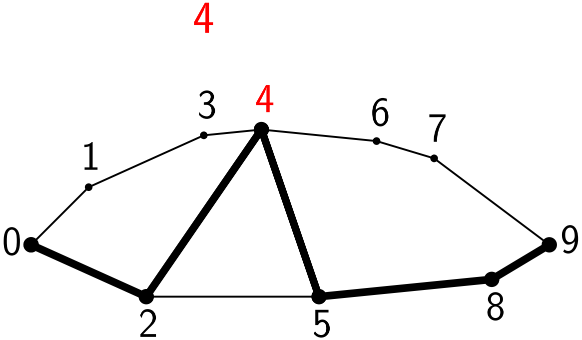

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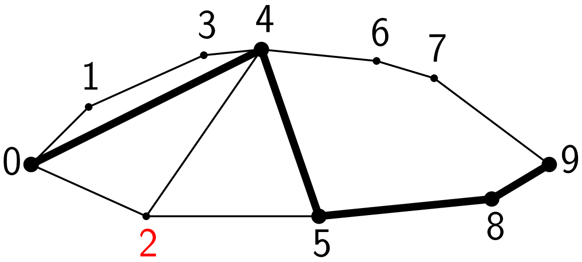

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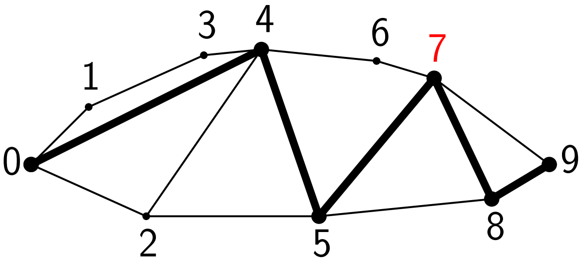

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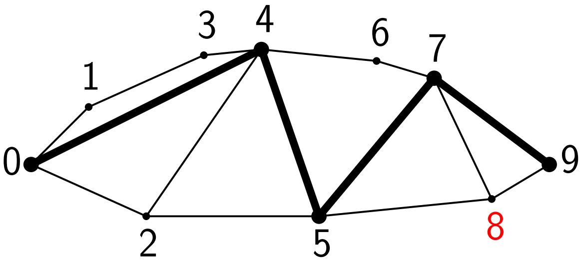

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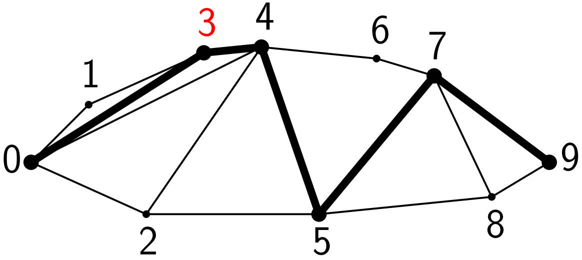

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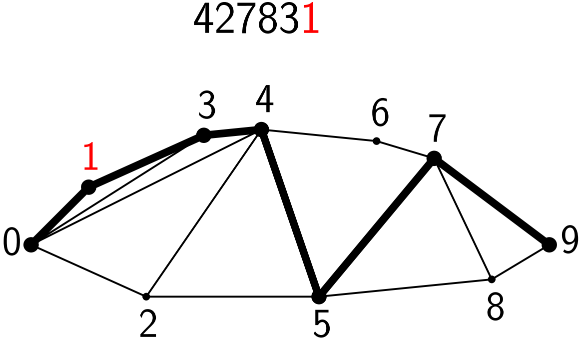

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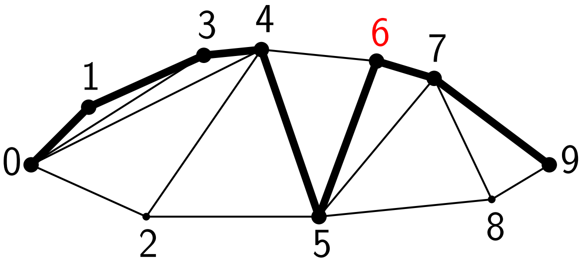

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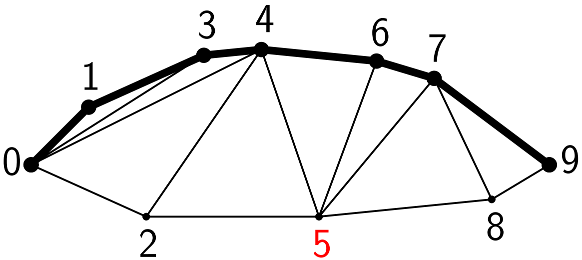

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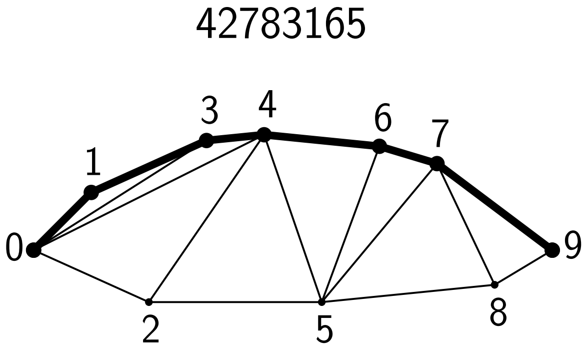
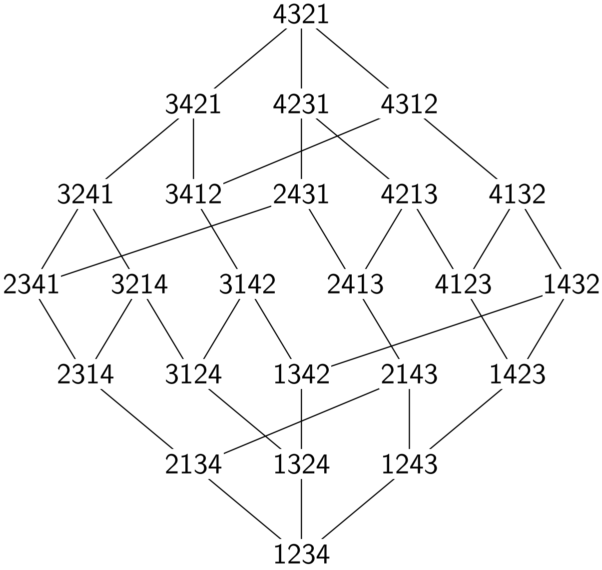


S_4 to triangulations



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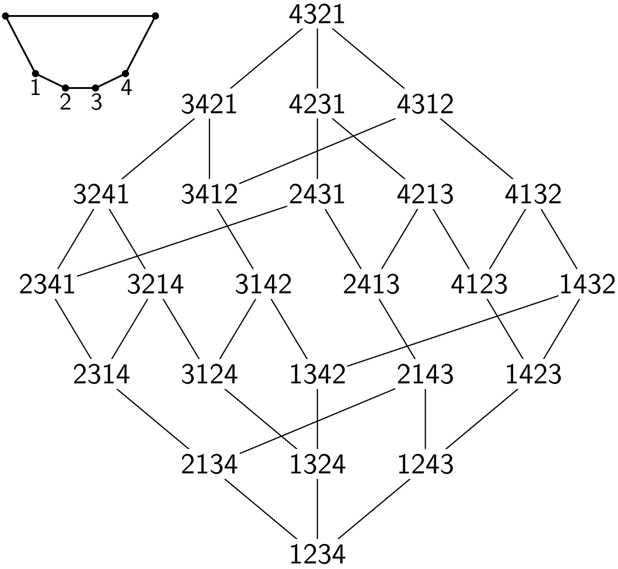

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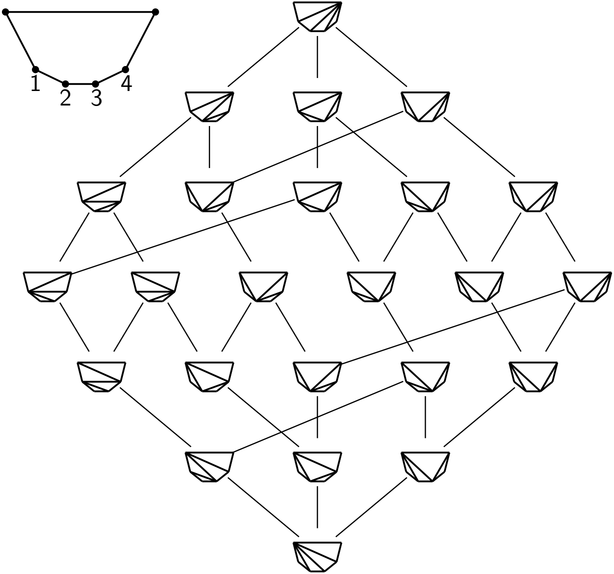
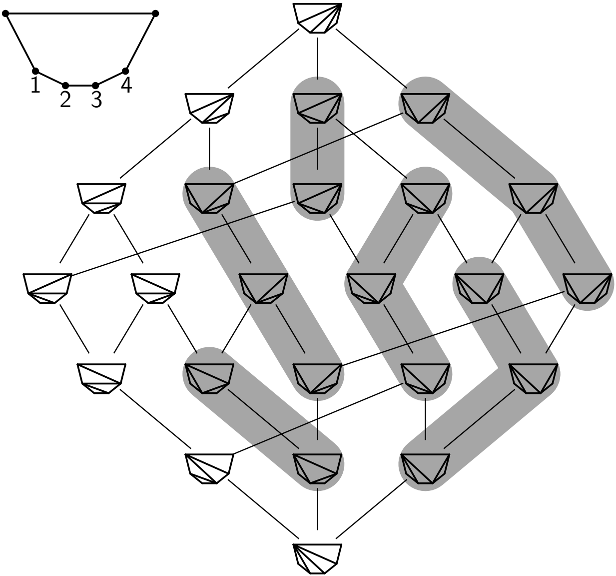


S_4 to triangulations   Fibers are a congruence.



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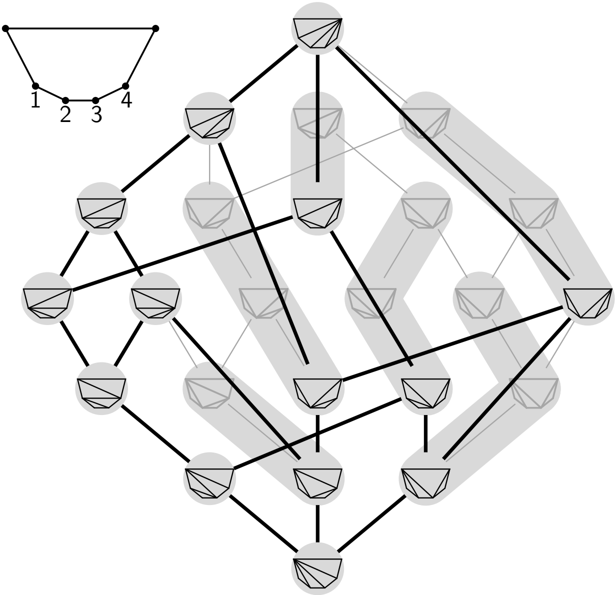

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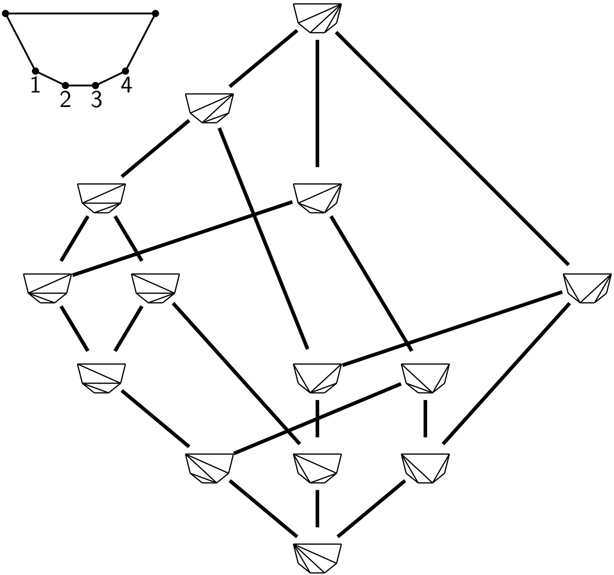
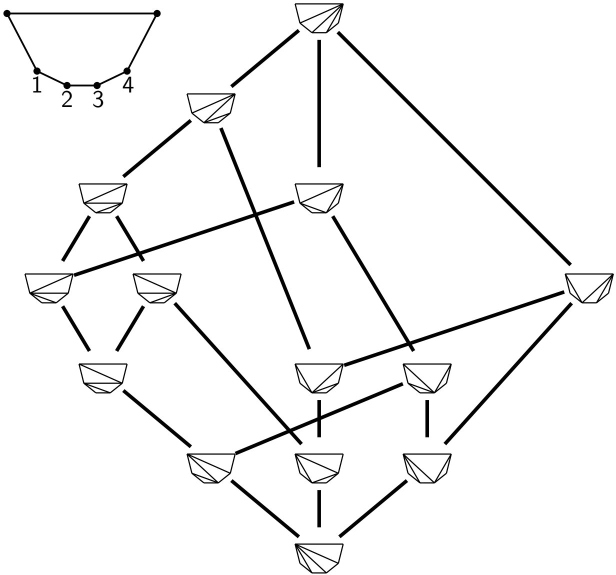
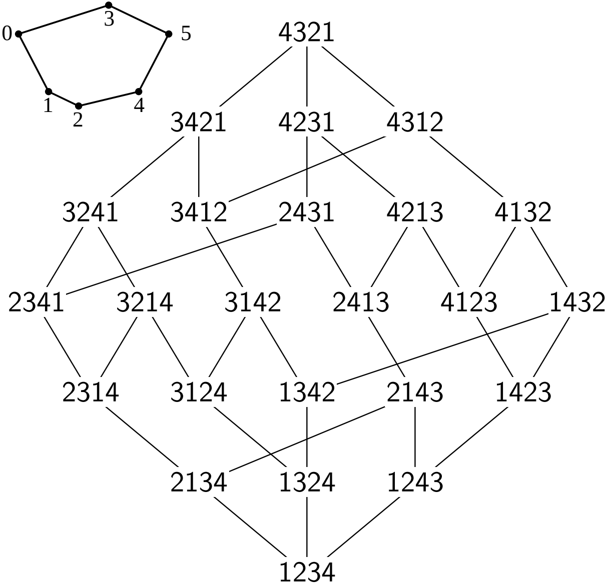


S_4 to triangulations

Fibers are a congruence. Quotient = Tamari lattice



S_4 to triangulations (for a different polygon)



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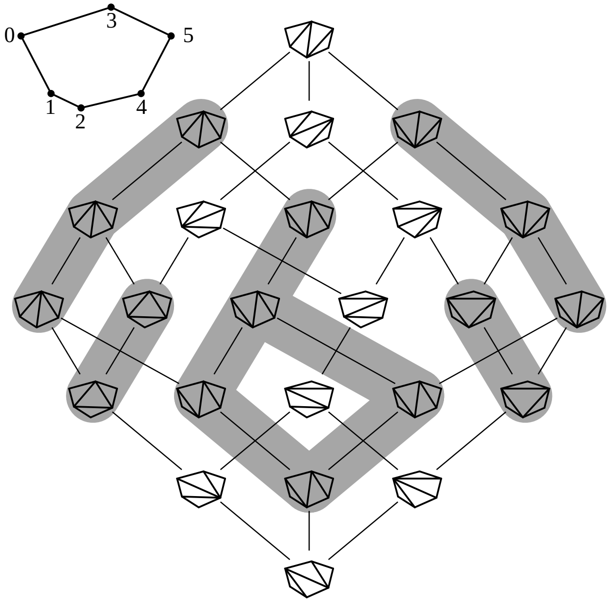

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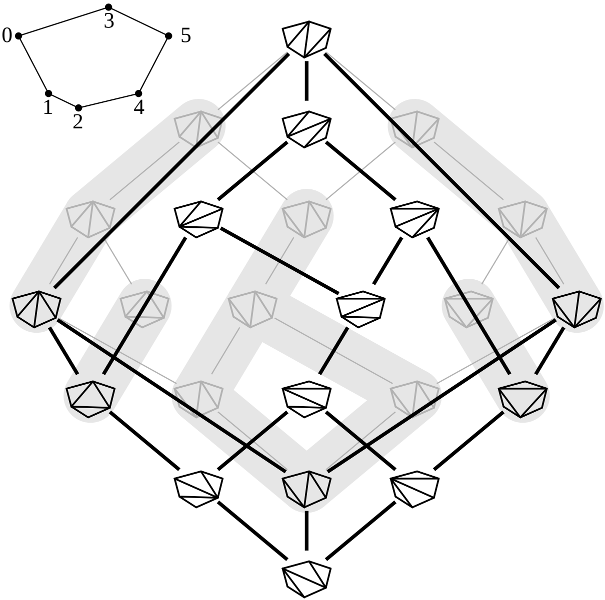

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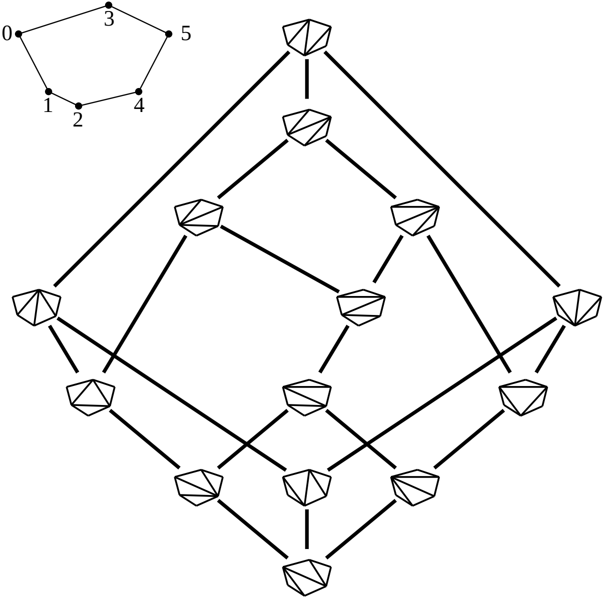
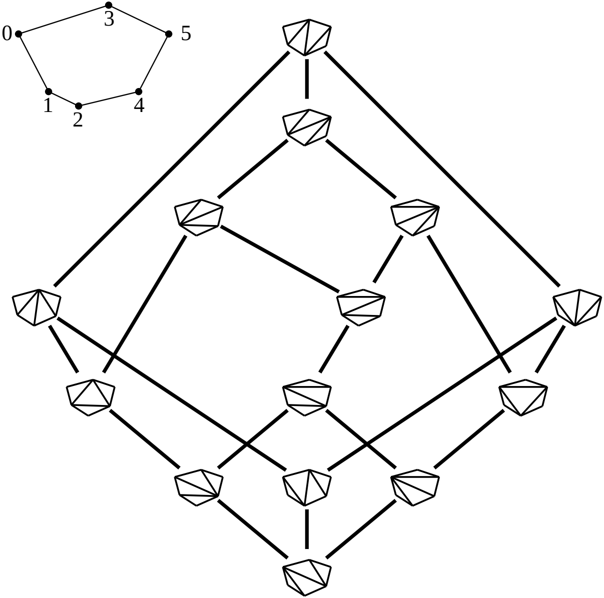


S_4 to triangulations (Quotient is a Cambrian lattice)



Order-theoretic characterization of congruence (concluded)

Recap of the example: We encountered a surjective map η from the weak order on permutations to the set of triangulations. One can check in general (using iterated fiber polytopes):

- Its fibers are intervals in the weak order.
- (ii) and (iii) hold for the fibers.
- Conclude: Fibers of η are a congruence $\equiv$, and η induces a lattice structure on S, isomorphic to $L/\equiv$.

In general, these lattices are "Cambrian lattices of type A." Covers are diagonal flips, and "going up" means increasing the slope of the diagonal. For a special choice of polygon, this is a Tamari lattice.

On finite L, an equivalence relation $\equiv$ is a lattice congruence iff:

(i) Each equivalence class is an interval in L.

(ii) The map $\pi_\downarrow$ is order-preserving.

(iii) The map $\pi^\uparrow$ is order-preserving.

Order-theoretic characterization of a lattice quotient

If L is a finite lattice and $\equiv$ is a congruence on L then the induced subposet $\pi_{\downarrow}L$ is a lattice, isomorphic to the quotient lattice $L/\equiv$.

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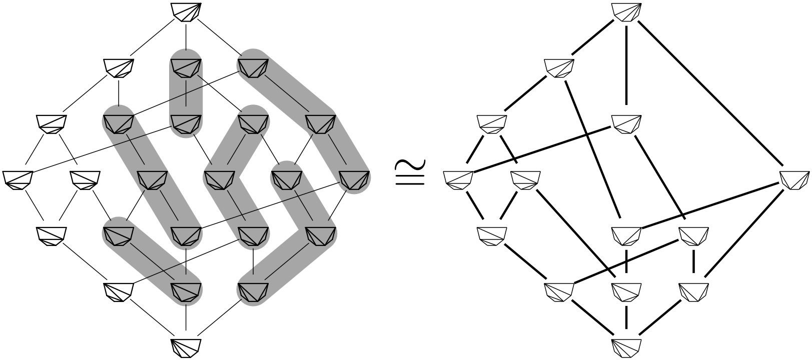
Example.



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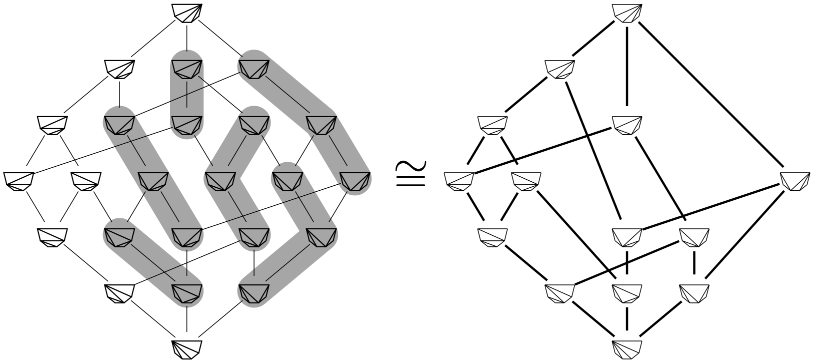


Caveat: $\pi_{\downarrow}L$ is a join-sublattice of L but can fail to be a sublattice:

Contracting edges

A congruence Θ contracts an edge $a \lessdot b$ if $a \equiv b$ modulo Θ.

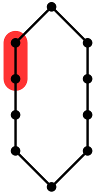
Since congruence classes are intervals, we can describe a congruence completely by specifying which edges are contracted.

As one might expect, edges cannot be contracted independently.

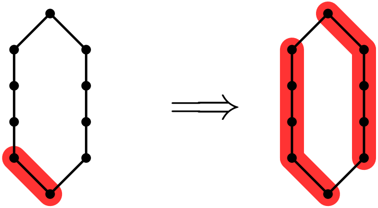
Say $a \lessdot b$ forces $c \lessdot d$ and write $(a \lessdot b) \to (c \lessdot d)$ if every congruence contracting $a \lessdot b$ also contracts $c \lessdot d$.

Example: Forcing in a polygon

A “side” edge can be contracted independently.

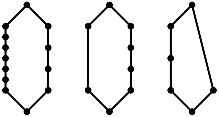


A “bottom” edge forces all side edges and the opposite “top” edge.



Dually, a “top” edge forces all side edges and the opposite “bottom” edge.

Polygonal lattices

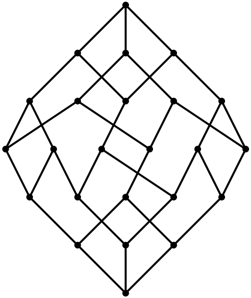
A polygon in a lattice: an interval like  etc.

L may have many polygons or none. It is called polygonal if it has as many polygons as possible. That is:

(i) If distinct elements y_1 and y_2 both cover an element x, then $[x, y_1 \vee y_2]$ is a polygon.

(ii) If an element y covers distinct elements x_1 and x_2, then $[x_1 \wedge x_2, y]$ is a polygon.

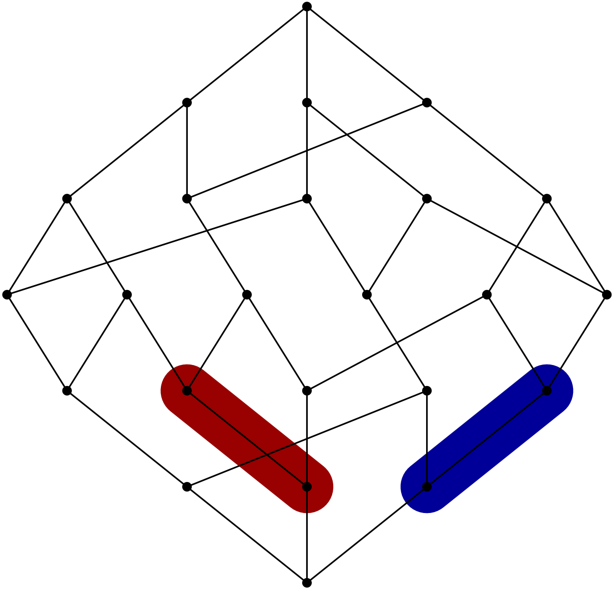
Example.



Proposition. The forcing relation in a polygonal lattice L is the transitive closure of the forcing relation in each polygon of L.

Example of forcing in a polygonal lattice

The congruence generated by contracting the **red** and **blue** edges.



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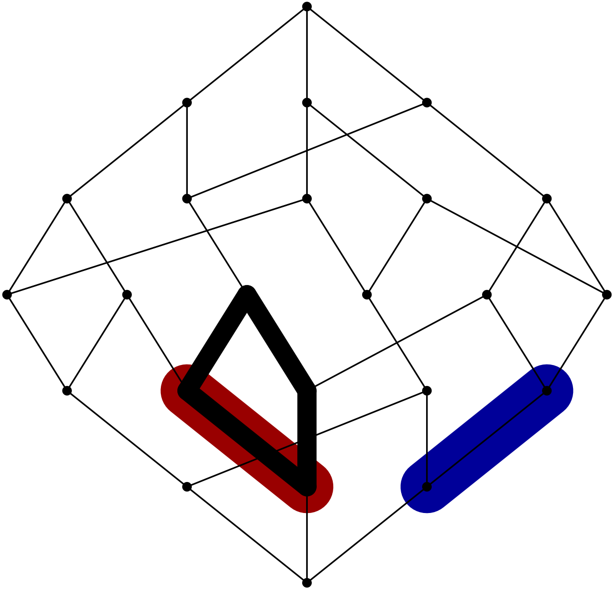

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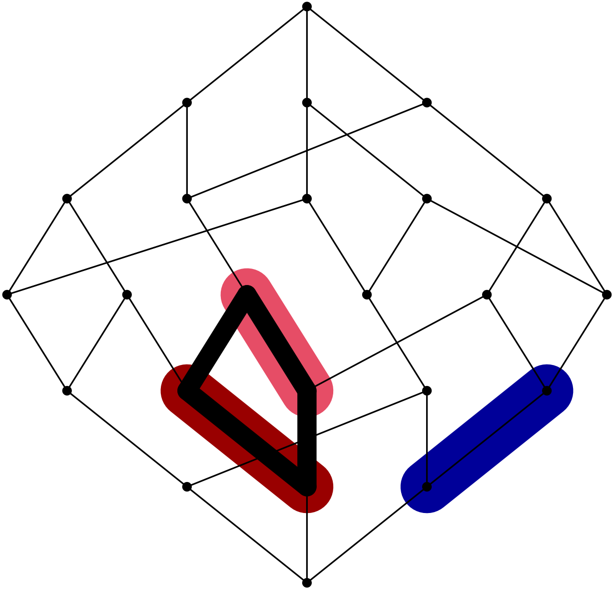

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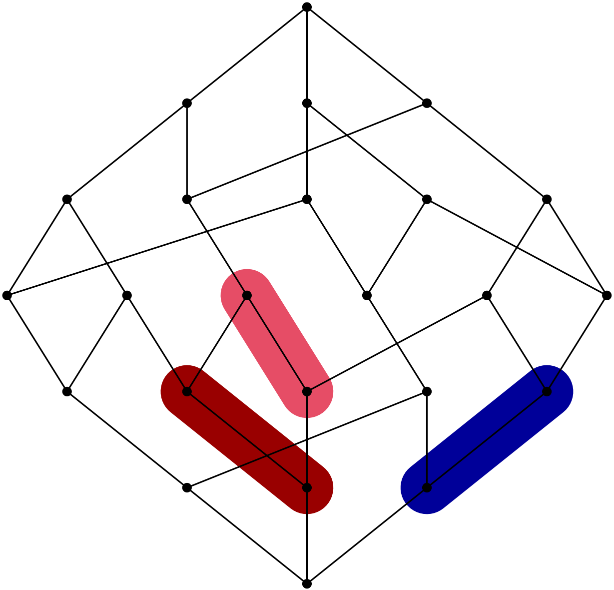

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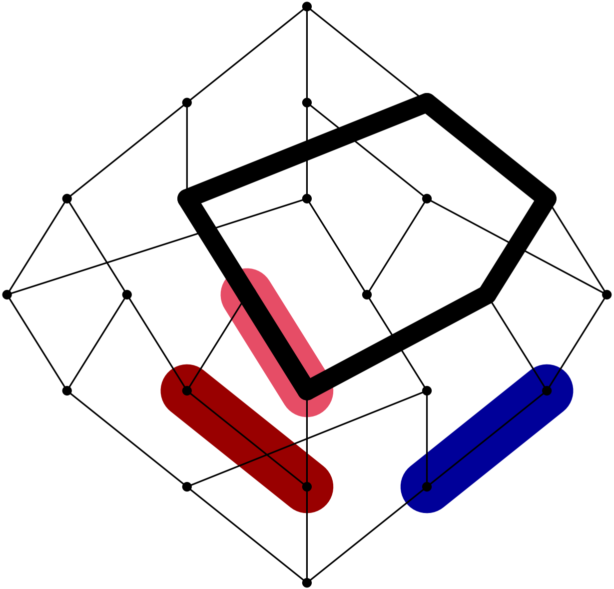

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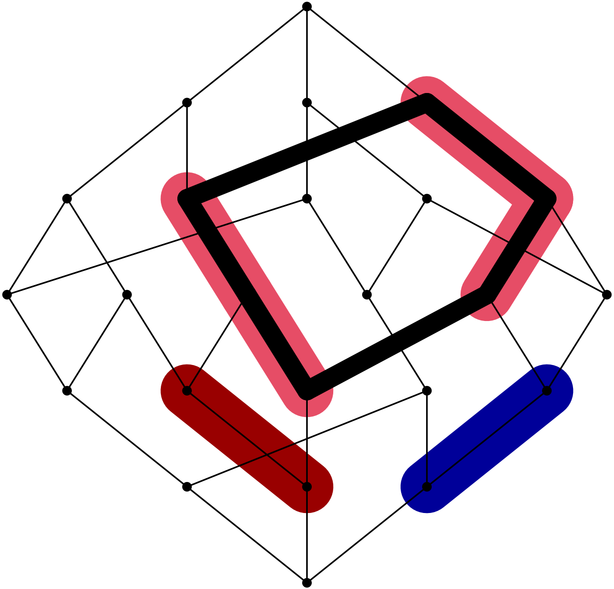

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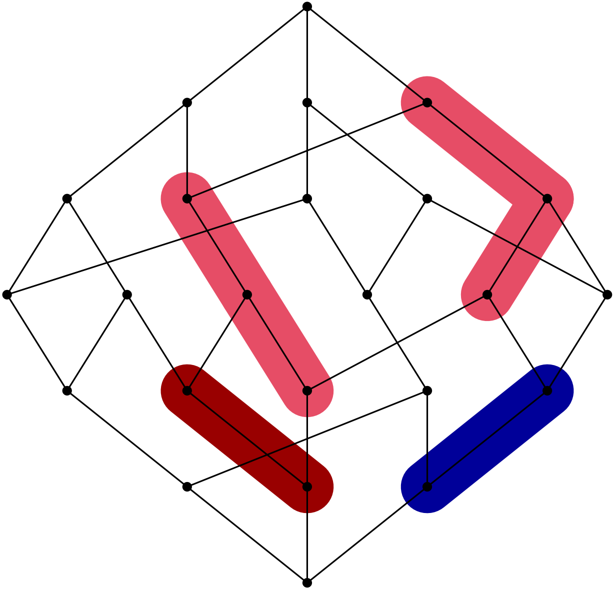

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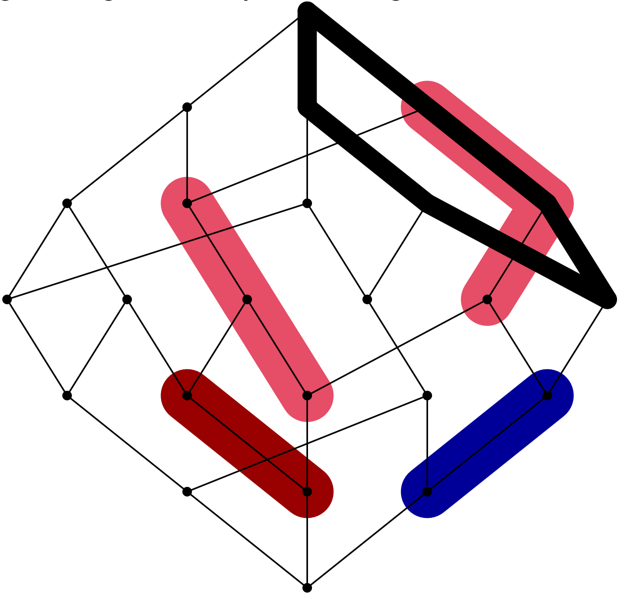

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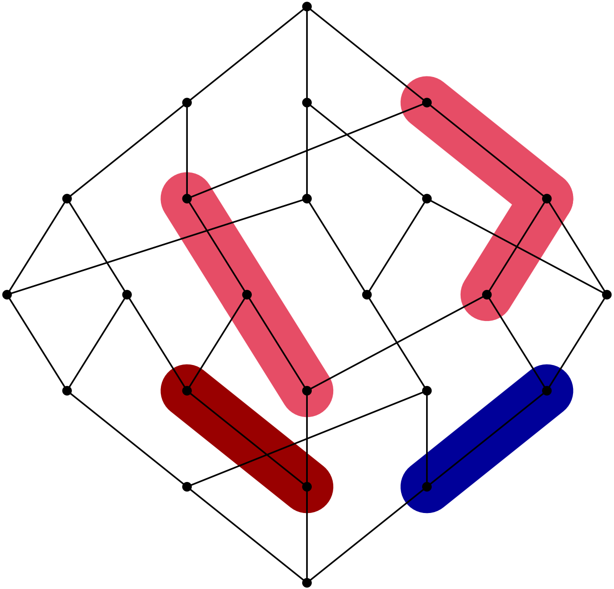

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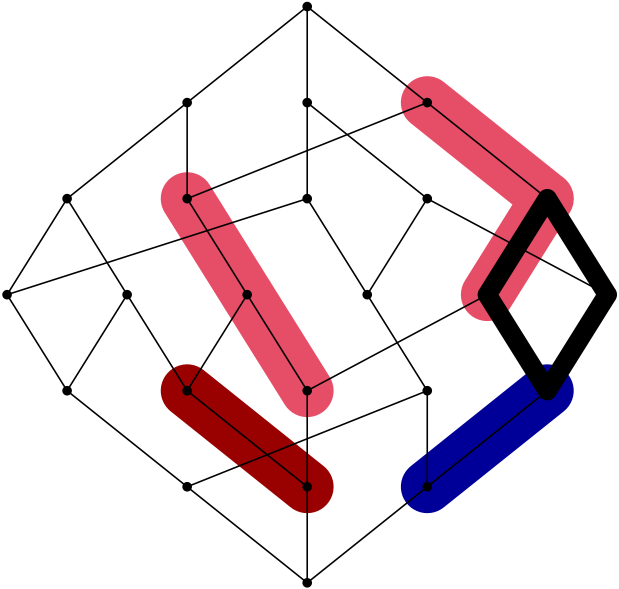

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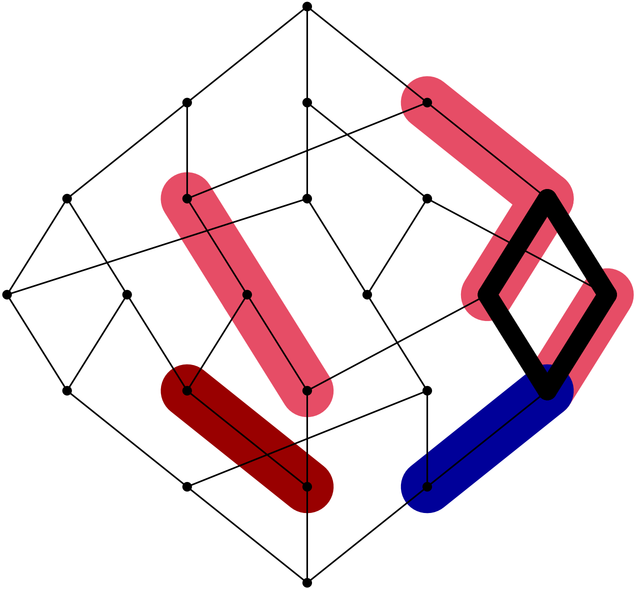

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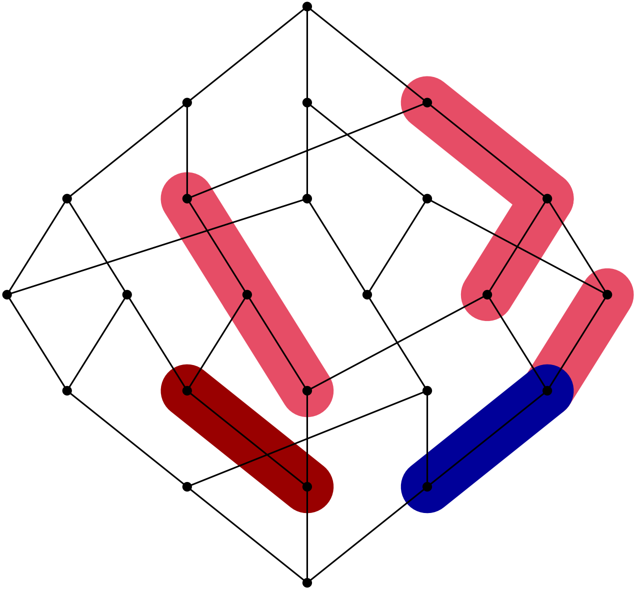

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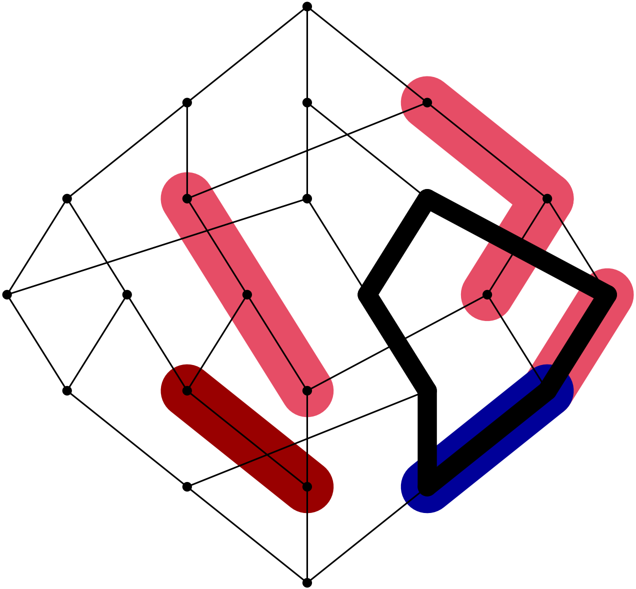

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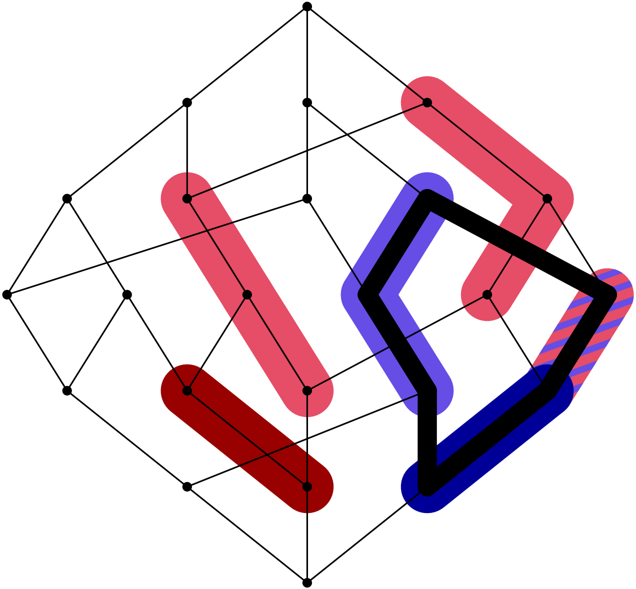

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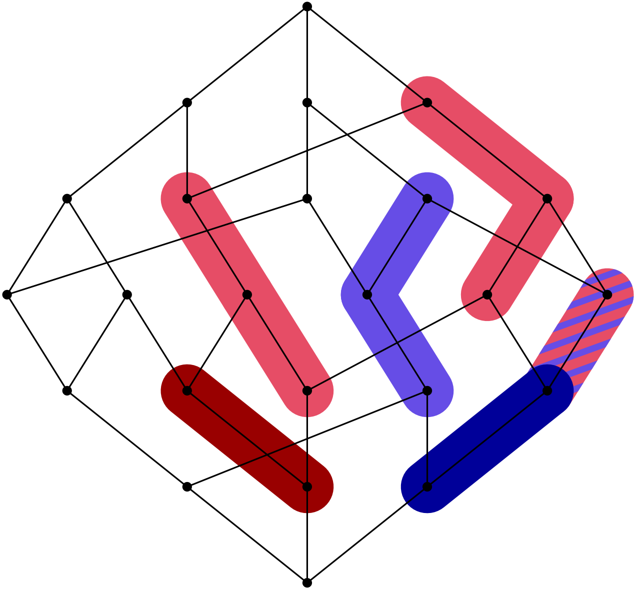

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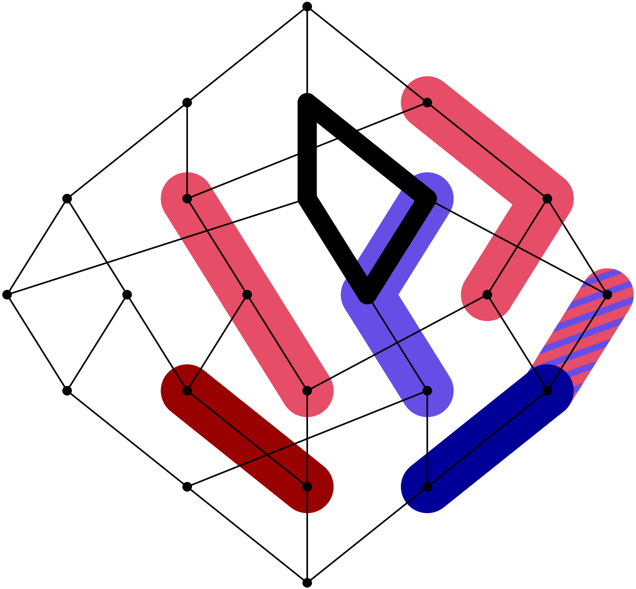

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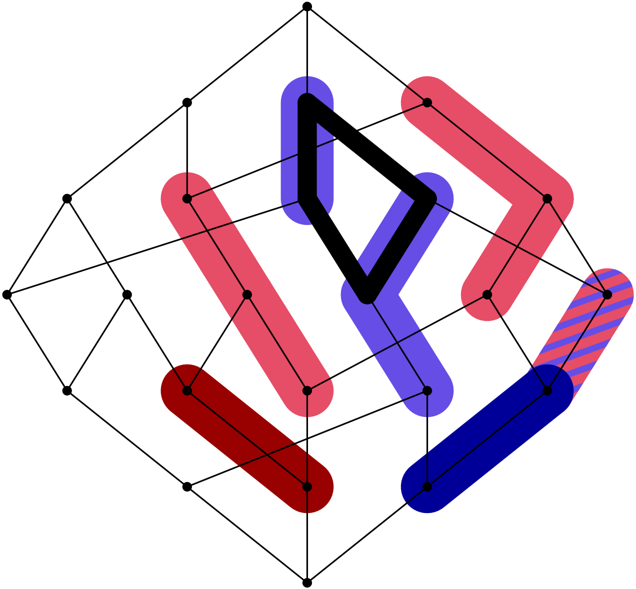

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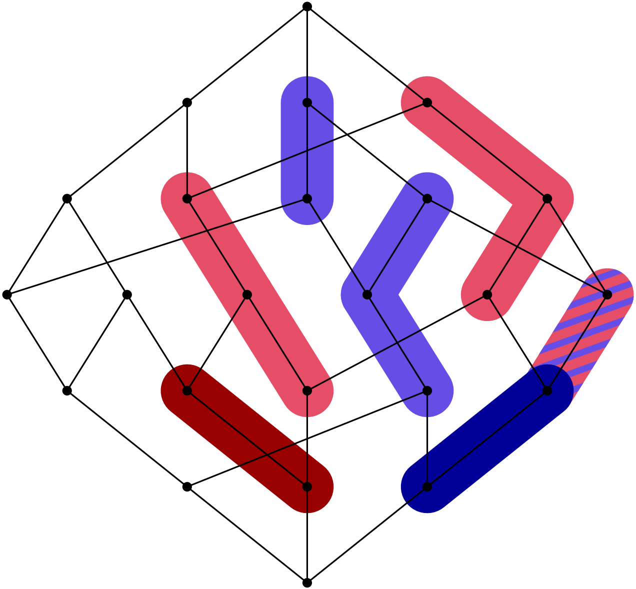

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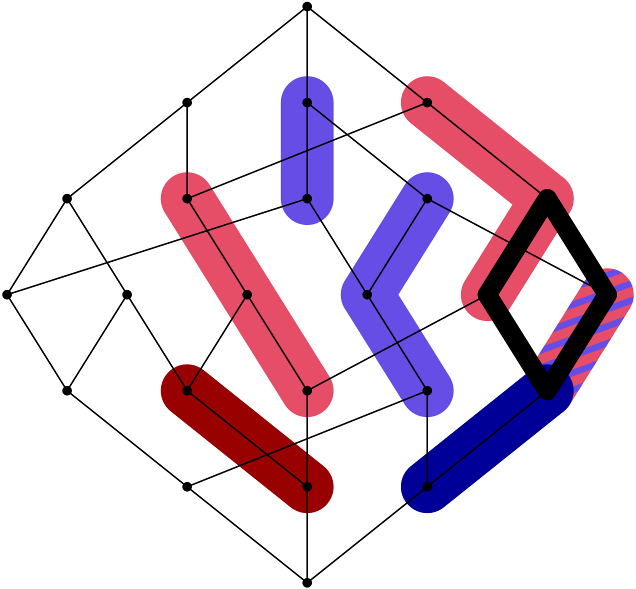

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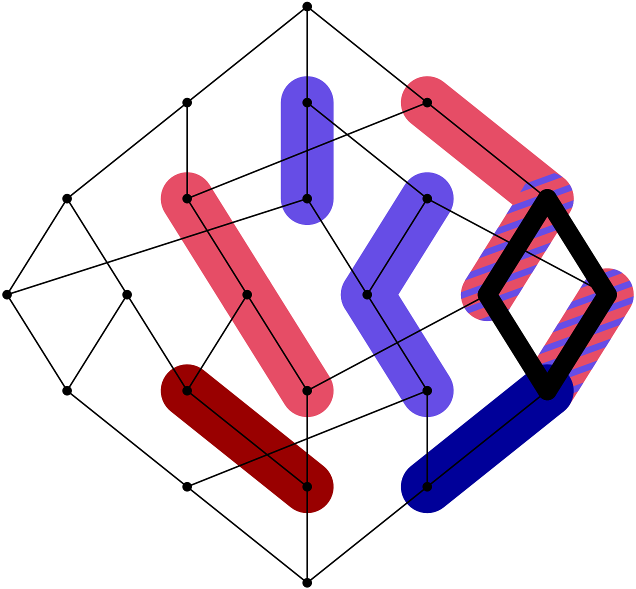

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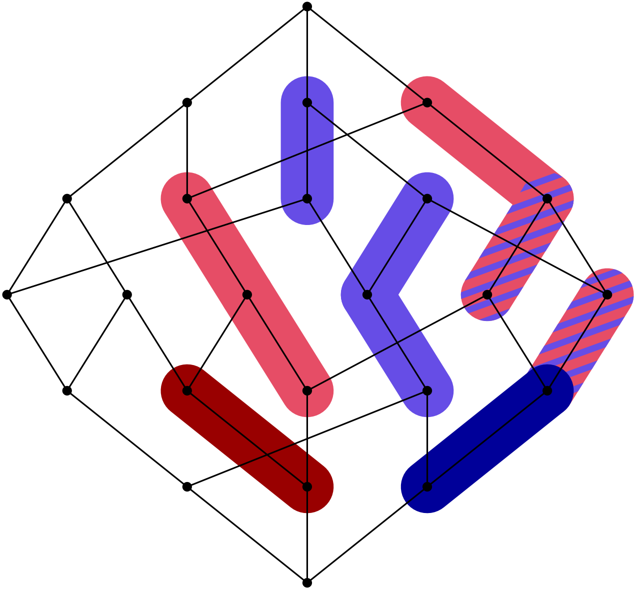


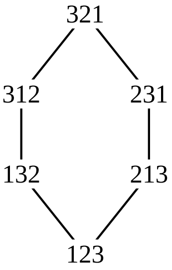
Section 2: Congruences on the weak order

The weak order

We write our permutations in one-line notation, e.g. 37284615.

The weak order on permutations is the partial order on permutations defined by these cover relations: Going "up" by a cover means putting adjacent entries out of numerical order.

Example. The weak order on S_3:

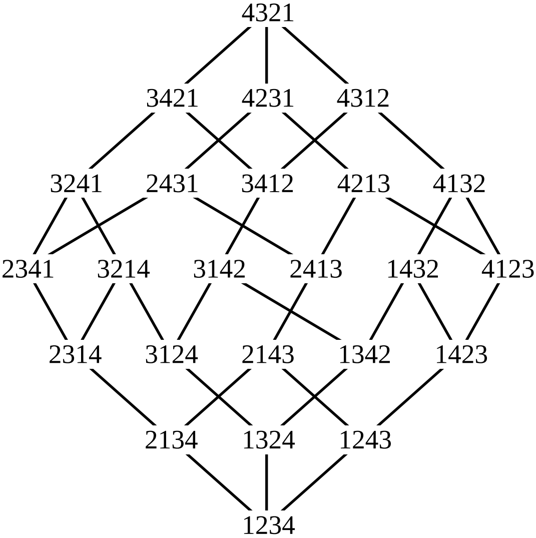


The symmetric group S_n is a finite Coxeter group (finite group generated by orthogonal reflections).

The weak order on any finite Coxeter group is a polygonal lattice. In fact it is "congruence uniform" and therefore "semidistributive".

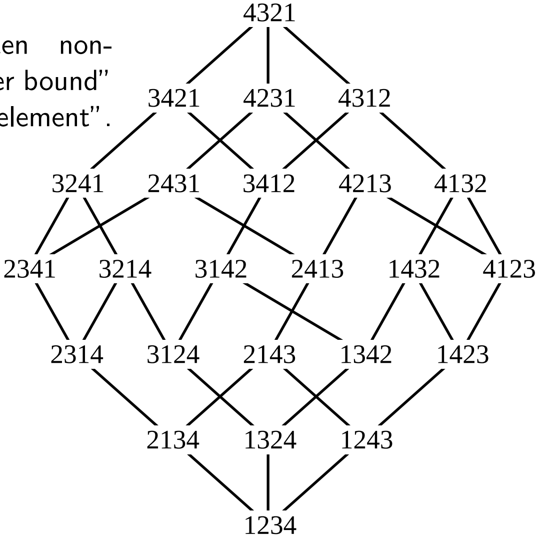
(We'll focus on S_n.)

Weak order on S_4



Join-irreducible elements of the weak order

Join-irreducible
= "can't be written non-trivially as a least upper bound"
= "covers exactly one element".



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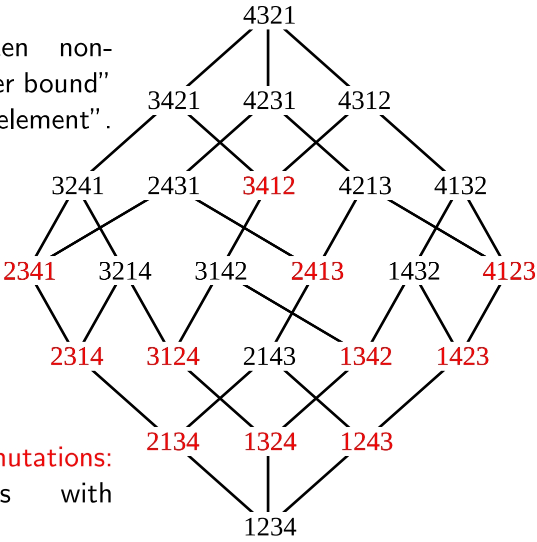

Join-irreducible permutations: Those permutations with exactly one descent.

Forcing equivalence classes (in the weak order)

$(a \lessdot b) \equiv (c \lessdot d)$ means $(a \lessdot b) \leftrightarrow (c \lessdot d)$

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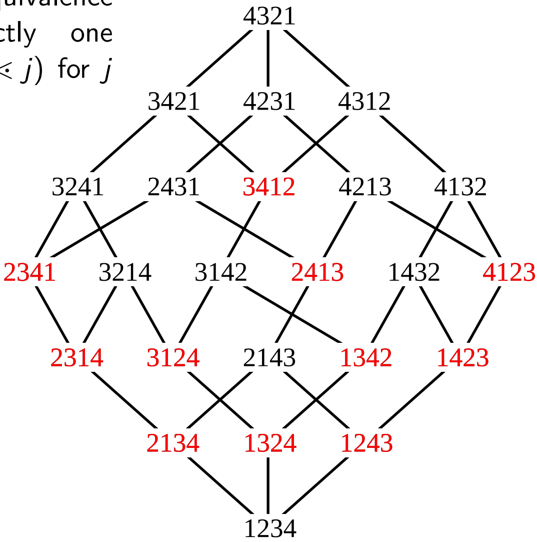

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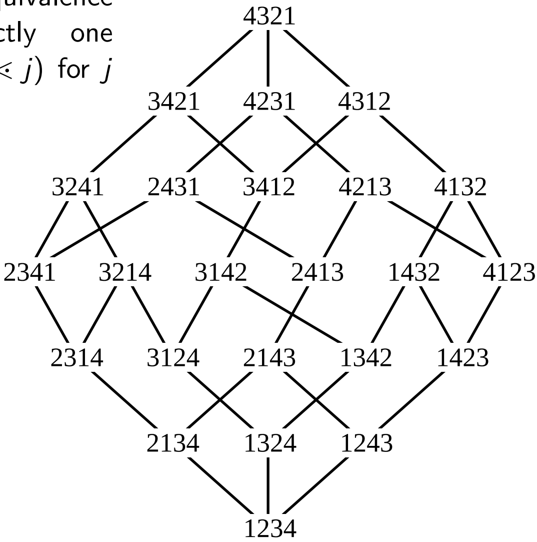

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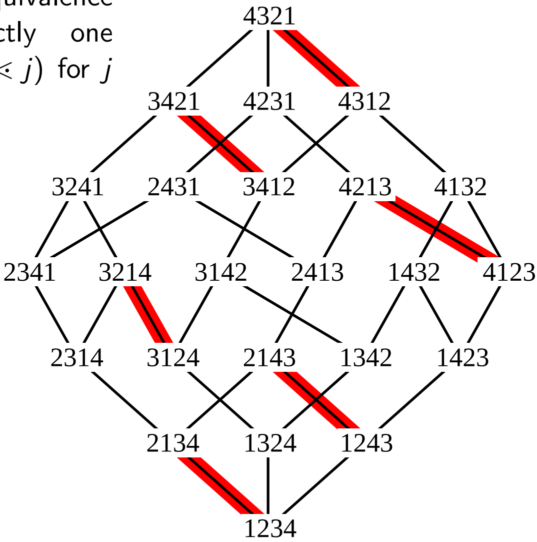

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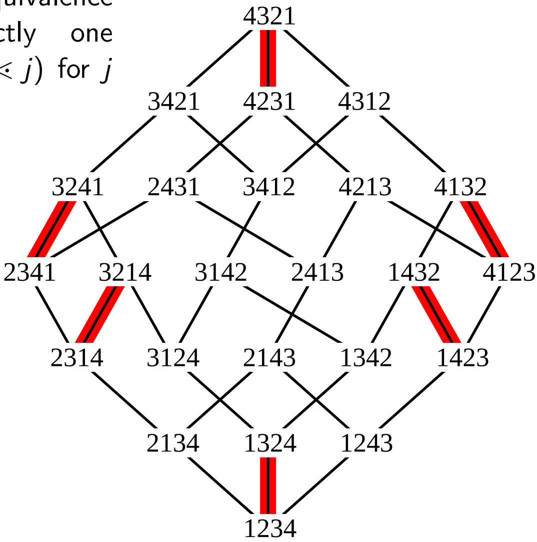

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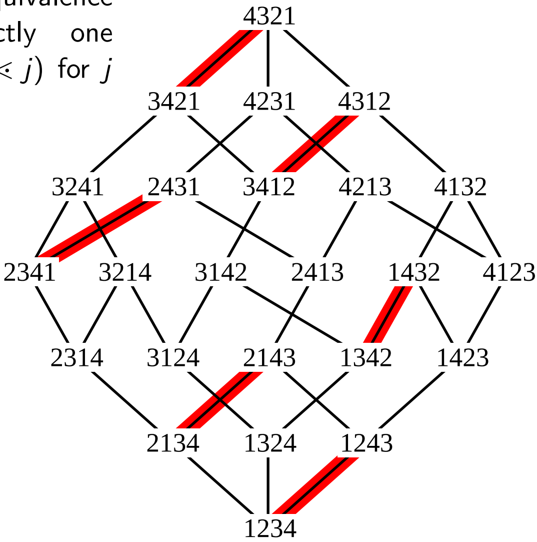

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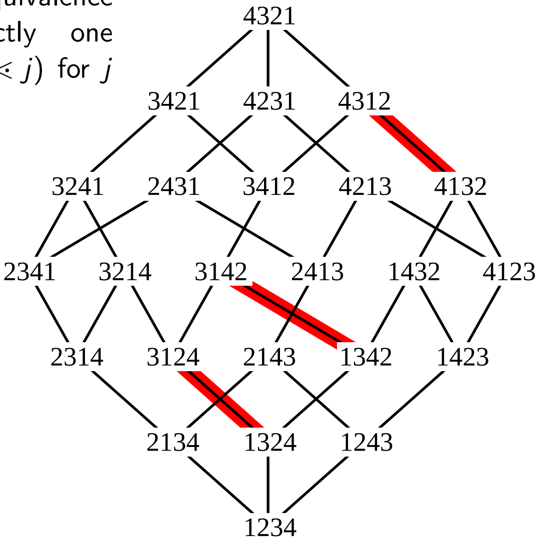

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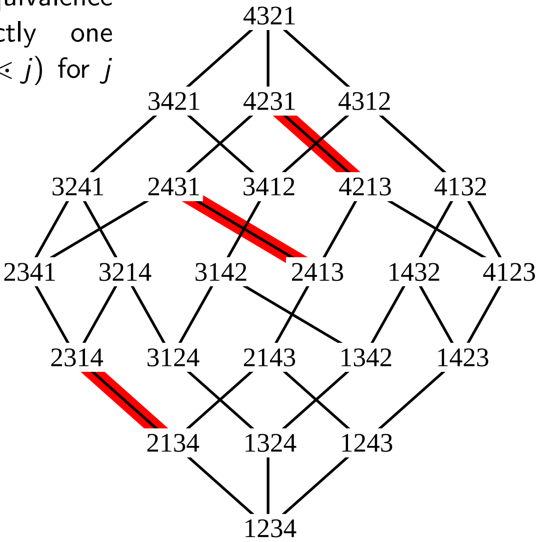

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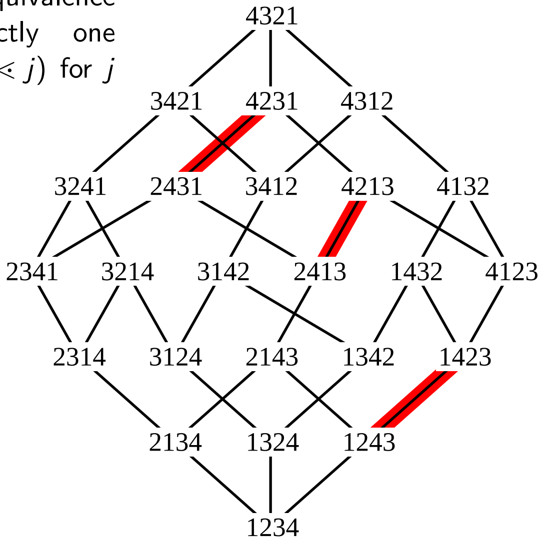

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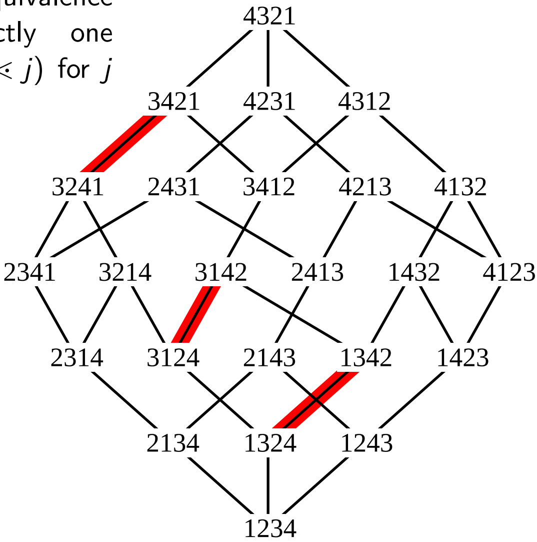

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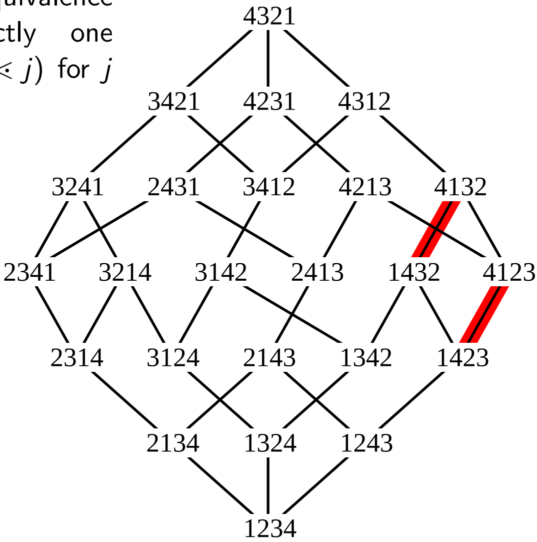

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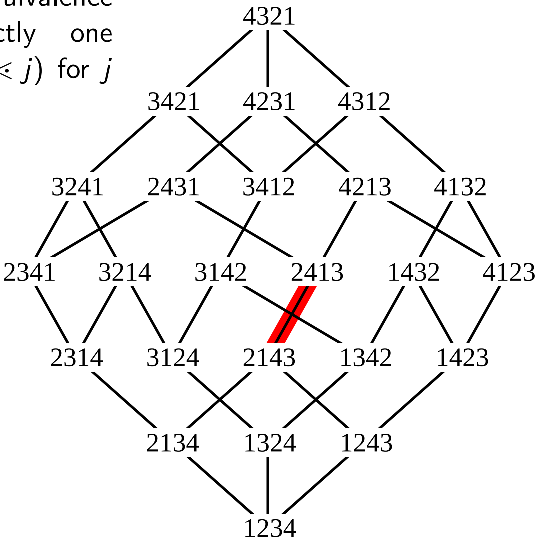

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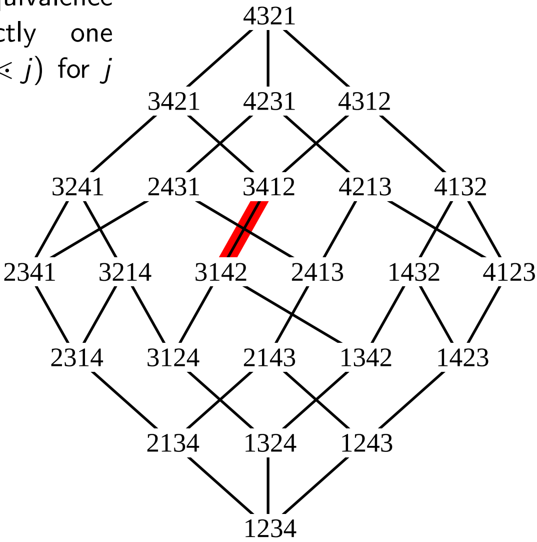

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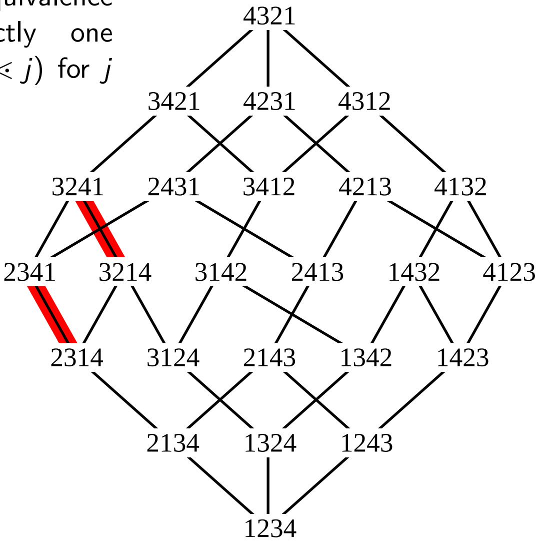

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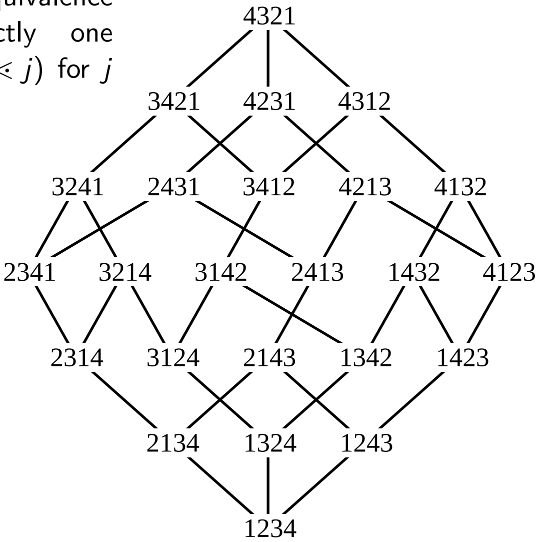

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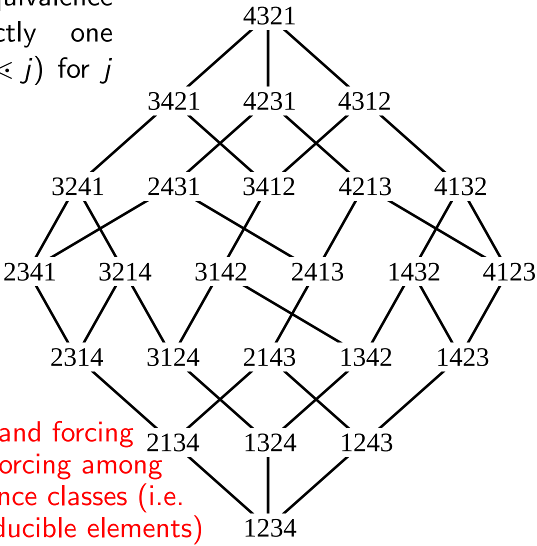


Upshot:
We can understand forcing by considering forcing among forcing equivalence classes (i.e. among join-irreducible elements)

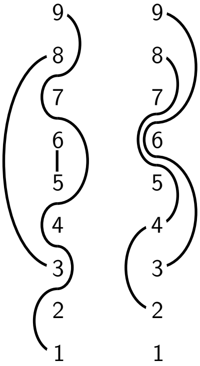
Noncrossing arc diagrams

Put $1, \ldots, n$ on a vertical line.

Arcs connect the point (monotone up/down). Consider arcs up to combinatorics (endpoints and what points it passes left/right of).

Compatibility of arcs: non-intersecting except possibly at their endpoints, and don't share the same upper endpoint or the same lower endpoint.

Noncrossing arc diagram: a collection of pairwise-compatible arcs.



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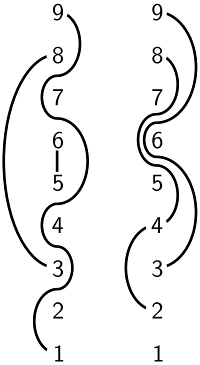

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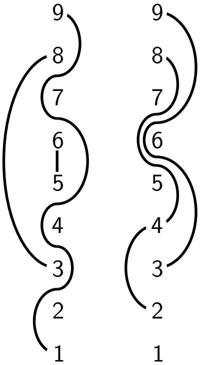


Seems like it should be difficult to count these...

In fact, there are $n!$ of them! We'll define a map δ such that:

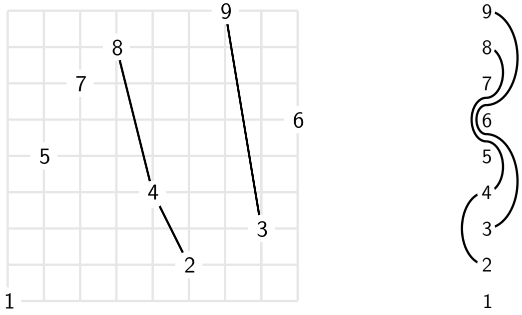
Theorem. The map δ is a bijection from permutations in S_n to noncrossing arc diagrams on n points. It restricts to a bijection between join-irreducible permutations and arcs.

Noncrossing arc diagrams (continued)

Constructing $\delta(\pi_1 \cdots \pi_n)$:

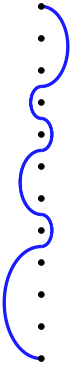
1. Graph $\pi_1 \cdots \pi_n$ by writing π_i at the point (i, π_i) in the plane.
2. Connect the descents with line segments.
3. Move the numbers to a single vertical line. Segments connecting descents become arcs.

Example. $\delta(157842936)$.



Forcing and subarcs

A subarc of a given arc:



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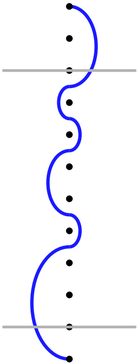

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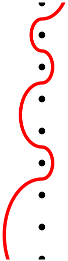

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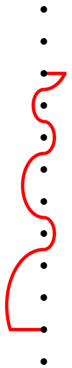

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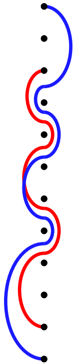

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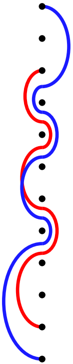


Key point: A join-irreducible permutation j_1 forces another join-irreducible permutation j_2 if and only if $\delta(j_1)$ is a subarc of $\delta(j_2)$.

Recall that a join-irreducible permutation serves as a representative of its forcing equivalence class. Recall also that arcs correspond to join-irreducible permutations. So we can understand lattice quotients of the weak order in terms of noncrossing arc diagrams and the subarc relation:

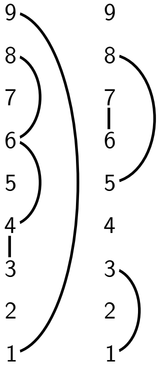
Constructing a lattice quotient of the weak order means choosing a set U of arcs that is closed under passing to subarcs.

$U \leftrightarrow$ forcing congruence classes that are not contracted.

The quotient is the subposet of the weak order induced by permutations π such that $\delta(\pi)$ only involves arcs in U.

Example: Right (or left-right) noncrossing arc diagrams

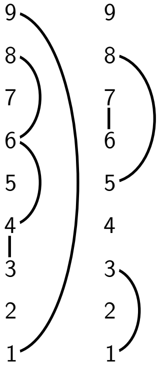
The set of all right arcs (arcs that don't pass left of any points) is closed under passing to subarcs. The quotient is the Tamari lattice. (Rotate diagrams 90° to get noncrossing partitions.)



Example: Right (or left-right) noncrossing arc diagrams

The set of all right arcs (arcs that don't pass left of any points) is closed under passing to subarcs. The quotient is the Tamari lattice. (Rotate diagrams 90° to get noncrossing partitions.)

Similarly left arcs don't pass right of any points, and there is a left-arc version of the Tamari lattice.



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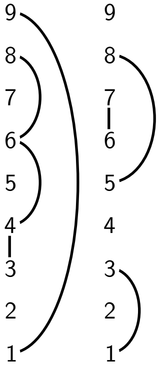

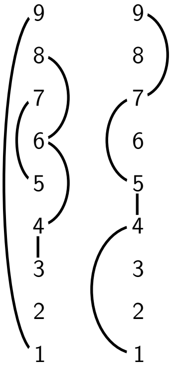

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The set {left arcs} $\cup$ {right arcs} is closed under passing to subarcs. A left-right noncrossing arc diagram is a diagram made from left and right arcs. (More on this quotient soon.)

What is a quotient of the weak order on permutations?

One answer: Choose a set U of arcs that will not be contracted by the congruence. (U is closed under passing to subarcs.) Elements of the quotient are noncrossing arc diagrams only using arcs in U.

Another answer: Think about arcs (forcing congruence classes) that will be contracted by the congruence. Use these to characterize permutations in the quotient by "(vincular) subsequence avoidance".

Suppose you have an arc α connecting i to j, with points $\ell_1, \ldots, \ell_k$ left of the arc and points $r_1, \ldots, r_m$ right of the arc.

What property of permutation π ensures that the arc diagram $\delta(\pi)$ does not use the arc α or any superarc of α?

π must not have a subsequence (for $j' \geq j$ and $i' \leq i$):

(permution of $\ell_1, \ldots, \ell_k$)$\cdot$(j' adjacent to i')$\cdot$(permution of $r_1, \ldots, r_m$)

Example (continued): Right noncrossing arc diagrams

You can get the set of all right arcs by leaving out all arcs connecting i to $i+2$ and passing to the left of $i+1$ (and of course leaving out all superarcs of these). The corresponding permutations avoid subsequences

$$(\geq i+2) \text{ adjacent to } (\leq i) \text{ followed by } i+1.$$

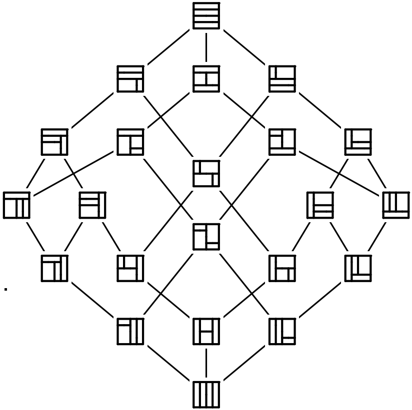
This quotient is the weak order restricted to 312-avoiding (also known as the Tamari lattice).

Example (continued): Left-right arc diagrams

You can get the set {right arcs} ∪ {left arcs} by leaving out the following arcs and their superarcs:

- arcs connecting i to $i+3$ and passing to the left of $i+1$ and right of $i+2$.
- arcs connecting i to $i+3$ and passing to the right of $i+1$ and left of $i+2$.

The corresponding permutations avoid the (vincular) patterns 2-41-3 and 3-41-2. They are the twisted Baxter permutations, counted by the Baxter number $\frac{2}{n^3+2n^2+n}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$. The quotient is isomorphic to a lattice of diagonal rectangulations (Law, R., 2011).



More examples

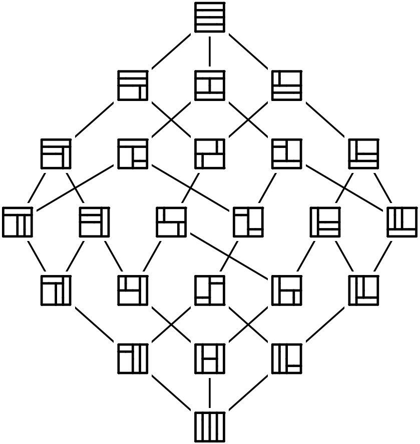
Cambrian lattices, given by a condition that interpolates between 231-avoidance and 312-avoidance. (For each point, we decide once and for all: either all arcs pass left of it or all arcs pass right of it.)

(Pilaud-Pons, 2016). Lattices of "permutrees" that interpolate between the weak order and Cambrian lattices. (Some points may allow arcs on both sides.)

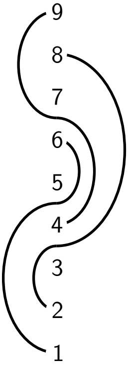
(Pilaud, 2015). A k-triangulation of a convex $(n+2k)$-gon is a maximal set of diagonals such that no $k+1$ of them are pairwise crossing. For each k, there is a lattice of k-triangulations, generalizing the Tamari lattice, appearing as a quotient of the weak order. (Each arc may pass left of at most k points.)

Another example: Generic rectangulations

There is a coarser congruence than the congruence associated to diagonal rectangulations that gives rise to a lattice of generic rectangulations.



The corresponding arc diagrams allow arcs that pass left of some points and right of some others, but only cross between left and right once.



Another example: Alternating arc diagrams

Alternating arcs: arcs that never pass left of two adjacent points and never pass right of two adjacent points.

The set of alternating arcs is closed under passing to subarcs, so there is an associated quotient of the weak order.

Theorem (E. Barnard, 2015, Châtel-Pilaud 2014).
There are $\binom{2n}{n}$ alternating arc diagrams on n points.

Section 3: Geometry and algebra

The geometry of Coxeter groups and congruences

Every Coxeter group W is a group generated by reflections.
$\mathcal{A}$: the collection of all reflecting hyperplanes for reflections in W.

For $W = S_n$, the reflecting hyperplanes are $x_i = x_j$ in $\mathbb{R}^n$.

Regions: connected components of the complement of $\mathcal{A}$.
These are in bijection with elements of W. Using the bijection, we see the weak order as a poset of regions.

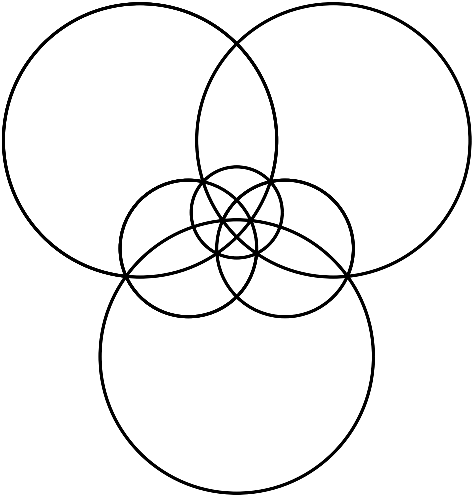
The regions and their faces constitute a simplicial fan.

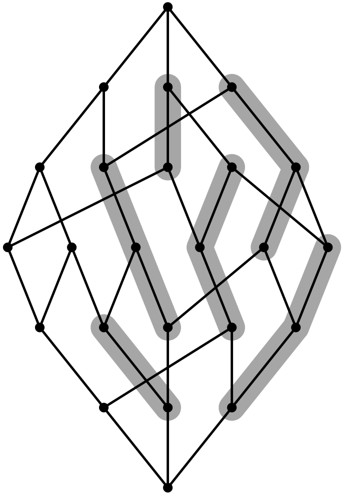
For any lattice congruence Θ on the weak order, define a collection $\mathcal{F}_\Theta$ of cones, closed under passing to faces:

Maximal cones of $\mathcal{F}_\Theta$ are unions, over congruence classes of Θ, of maximal cones of the fan defined by $\mathcal{A}$.

Theorem. $\mathcal{F}_\Theta$ is a fan.

Example: $\mathcal{F}_\Theta$ for a congruence on the weak order on S_4





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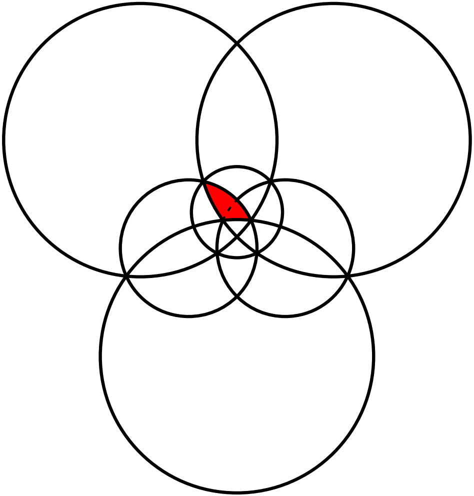

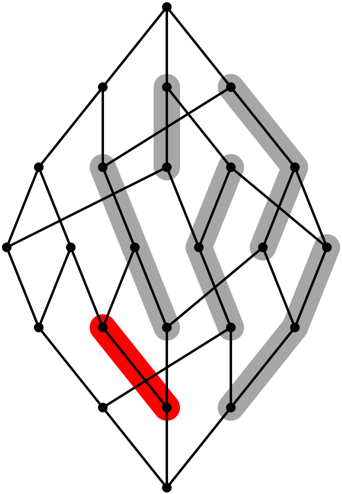

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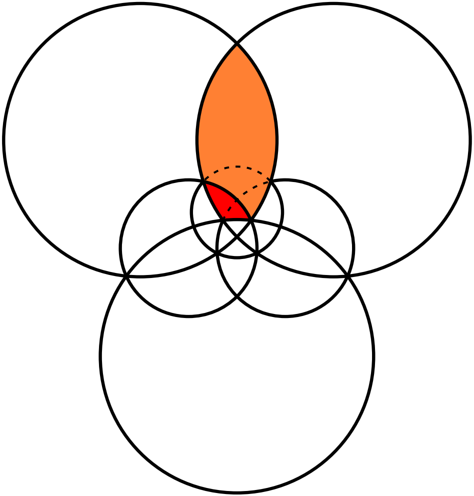

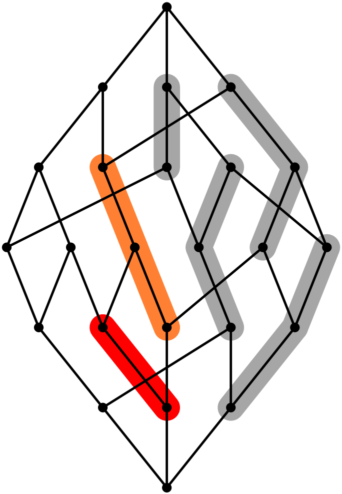

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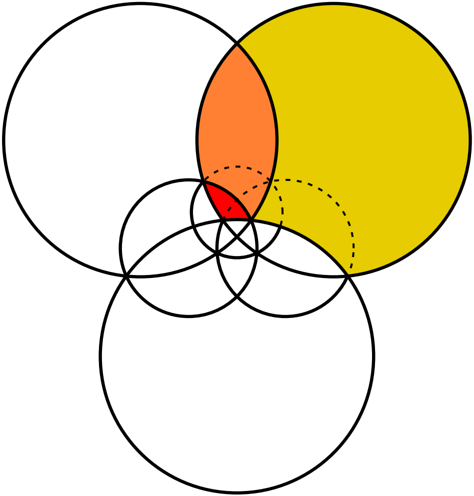

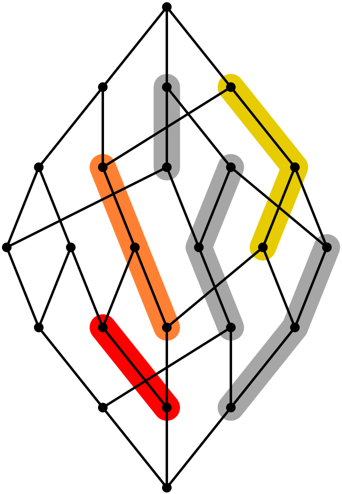

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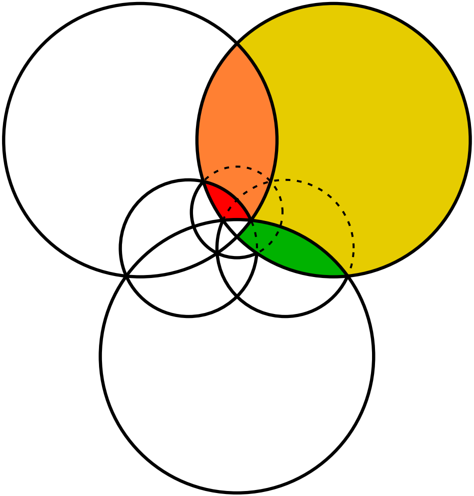

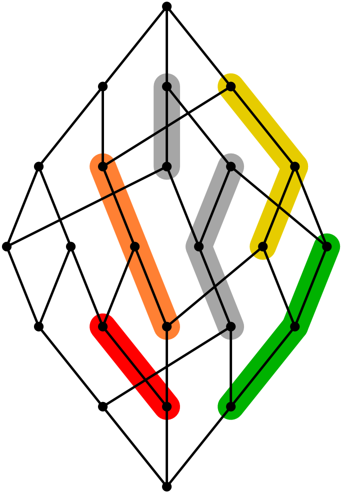

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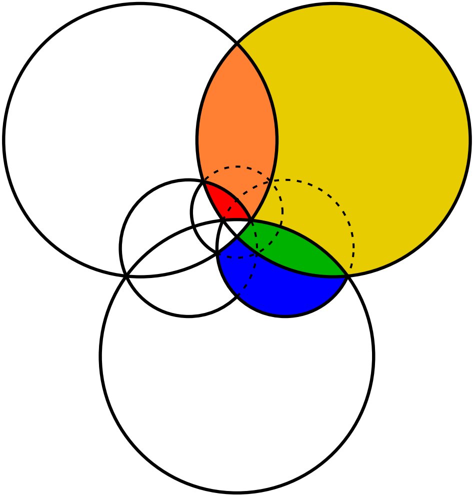

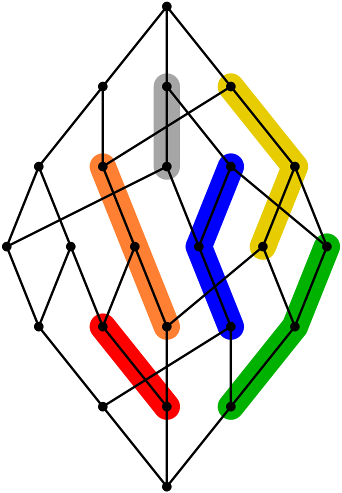

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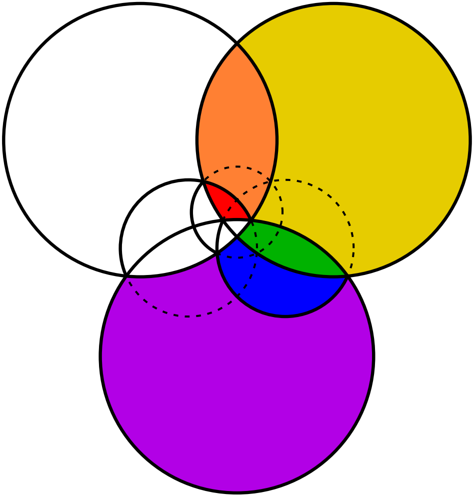

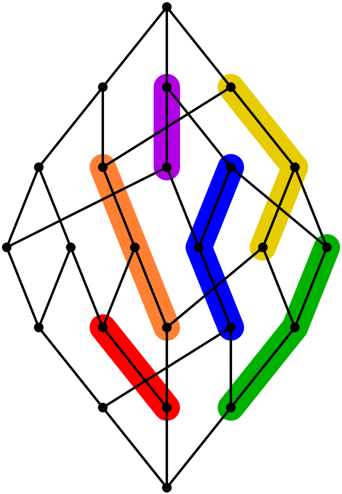

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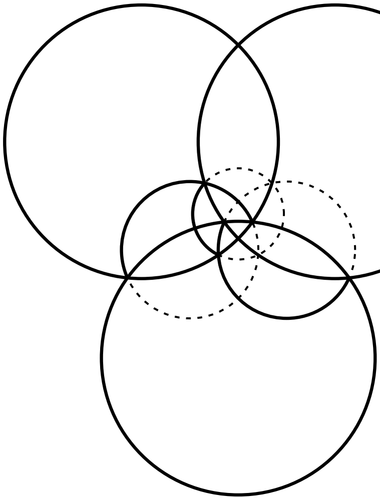
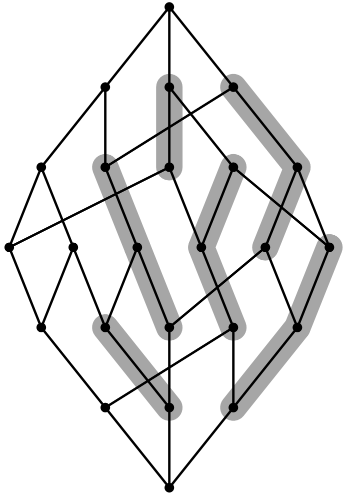
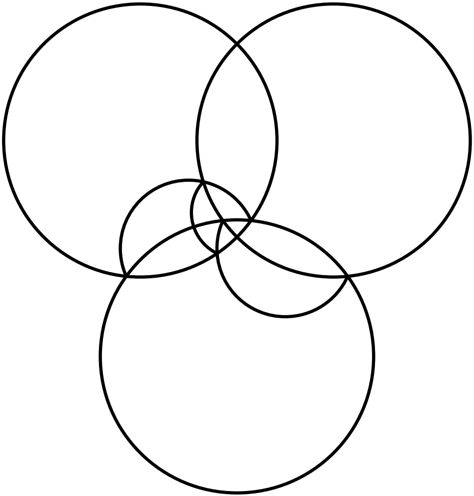
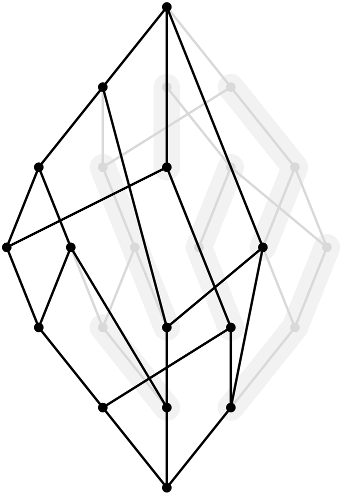




Example: $\mathcal{F}_\Theta$ for a congruence on the weak order on S_4

$\mathcal{F}_\Theta =$ normal fan of associahedron. $S_4/\Theta =$ Tamari lattice.





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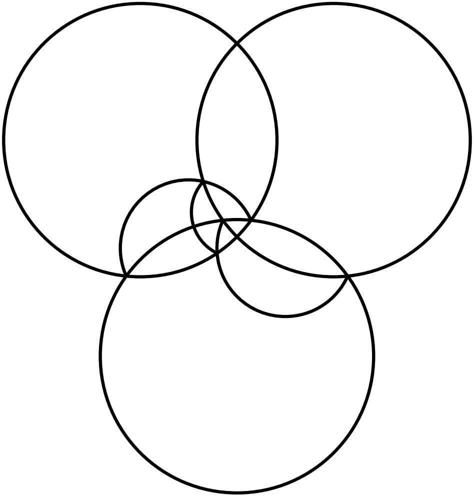
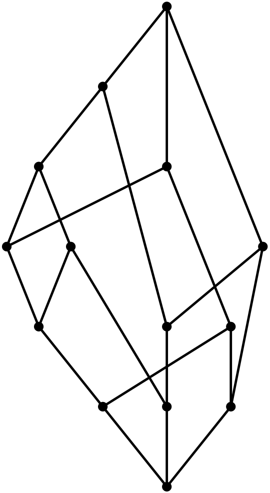




Polytopes, generalized associahedra, and cluster algebras

Theorem (Pilaud, Santos, 2018). For any congruence Θ on the weak order on S_n, the fan $\mathcal{F}_\Theta$ is the normal fan of a polytope.

This is probably true for any Coxeter group W, or more generally.

Theorem (R., Speyer, 2006). For any Coxeter group W and any Cambrian congruence, the Cambrian fan $\mathcal{F}_\Theta$ is combinatorially isomorphic to the normal fan of a generalized associahedron.

Theorem (Hohlweg, Lange, Thomas, 2006). The Cambrian fan is the normal fan of a certain realization of the generalized associahedron. (They gave an explicit realization.)

Theorem(R.-Speyer 2006, Yang-Zelevinsky 2008, R.-Speyer 2011). The Cambrian fan coincides with the **g**-vector fan of the associated cluster algebra.

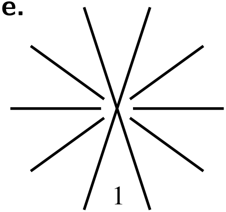
Shards

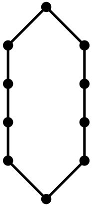
To make the fan $\mathcal{F}_\Theta$ for a congruence Θ on the weak order, we glue cones of the Coxeter fan together according to congruence classes.

So: contracting an edge means removing the wall between two adjacent cones.

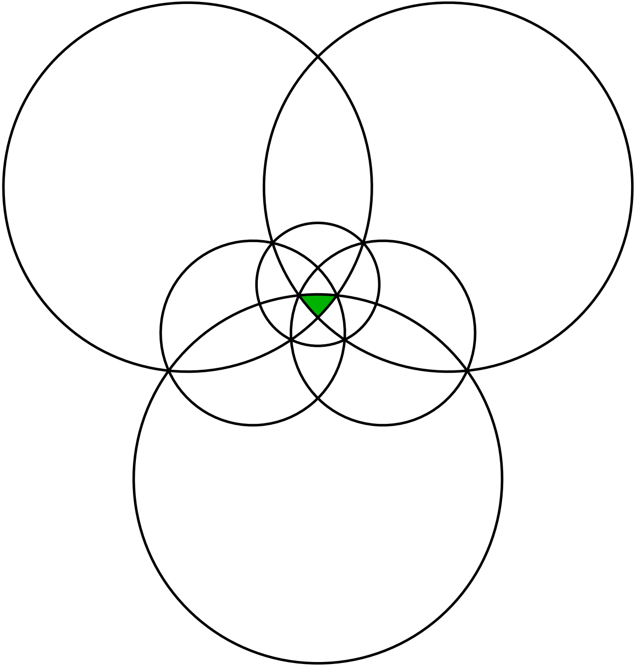
A shard is (the union of) the collection of walls corresponding to a forcing equivalence class. Each shard turns out to consist of walls all in the same hyperplane. There is a simple geometric description.

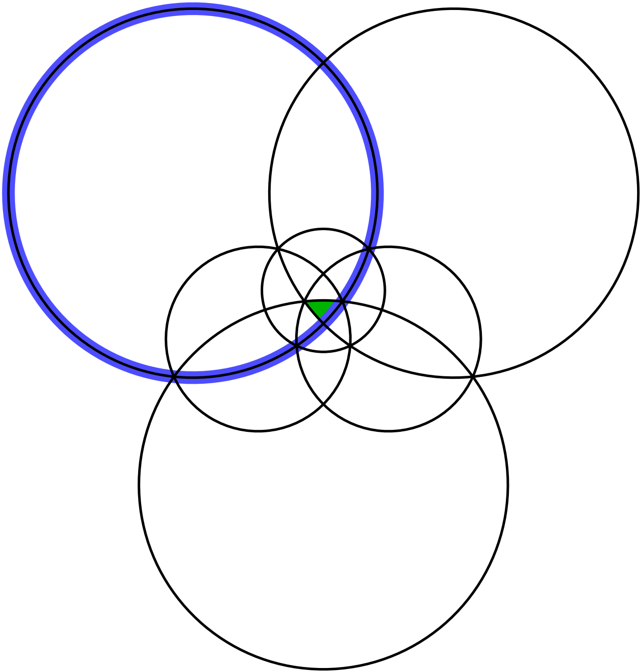
Example.

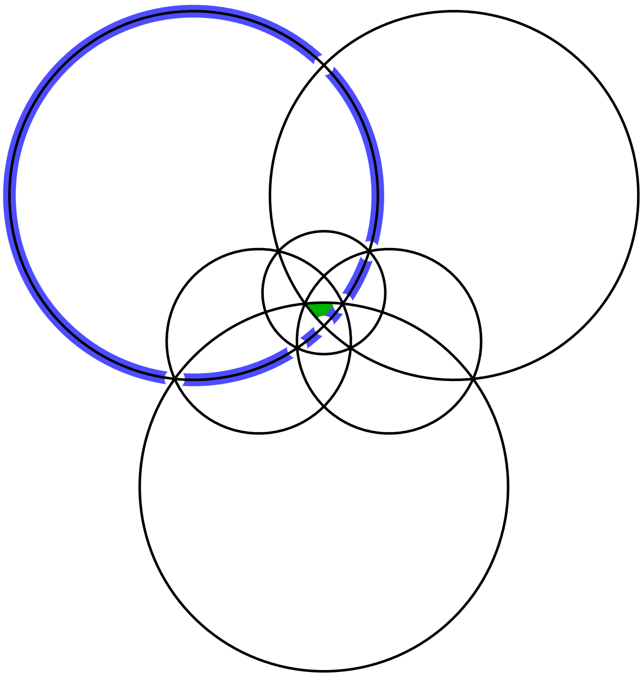


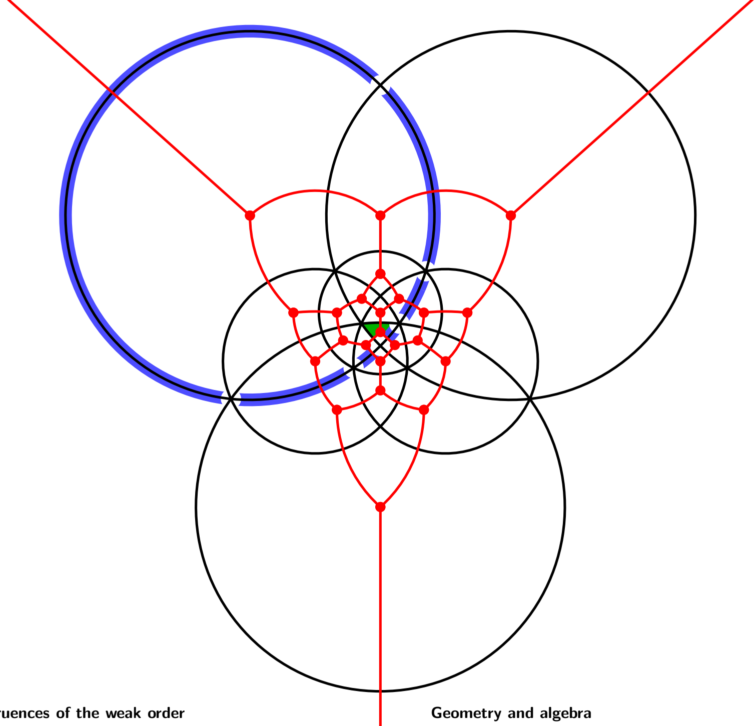


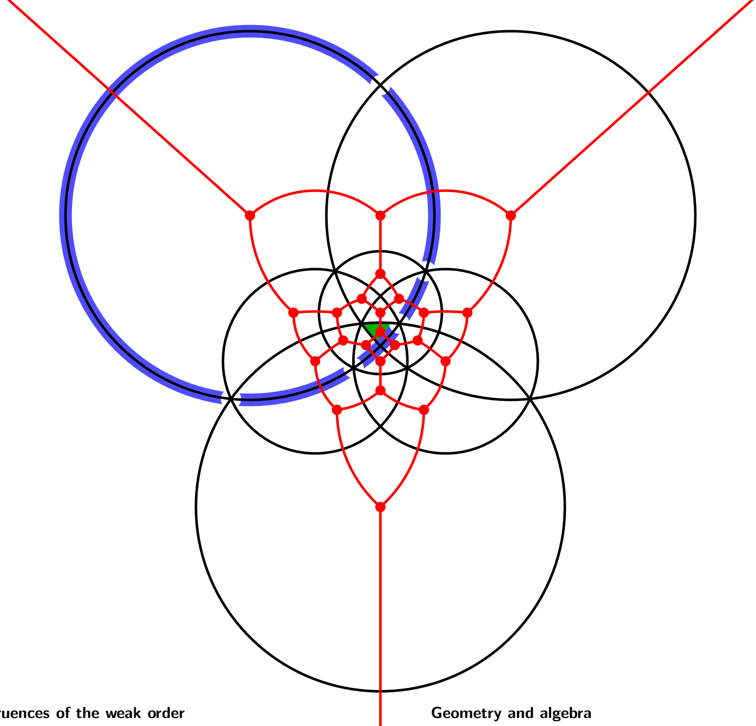
We describe a congruence by specifying which shards are removed. There is a geometric description of forcing among shards.

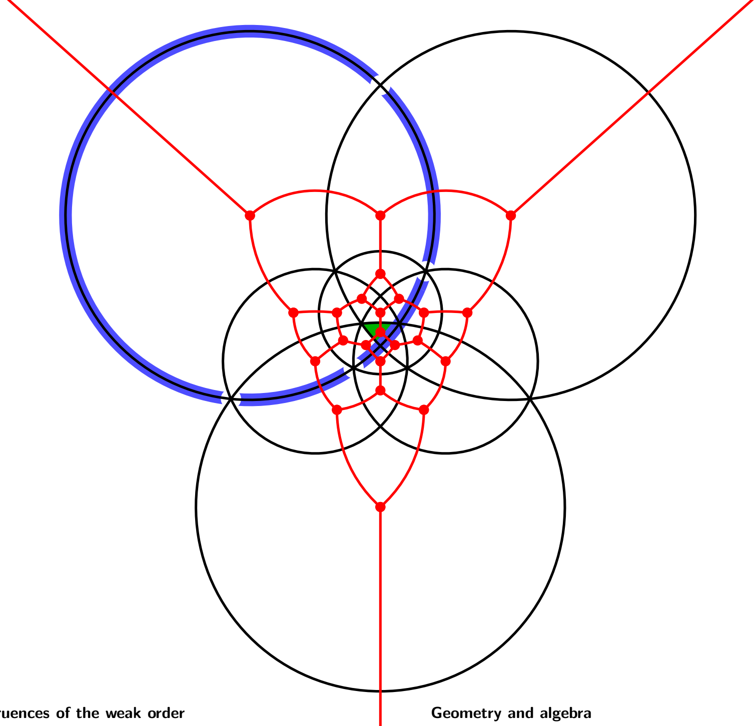


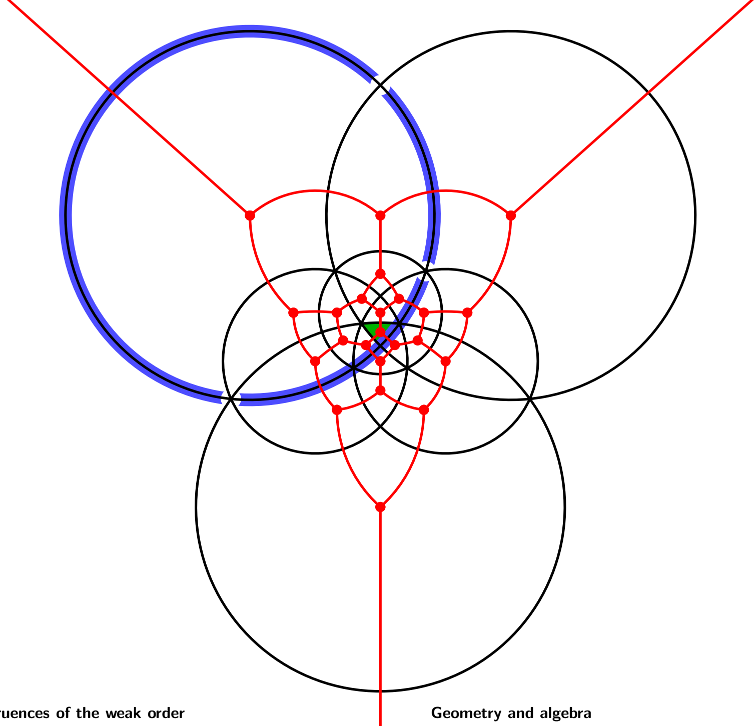


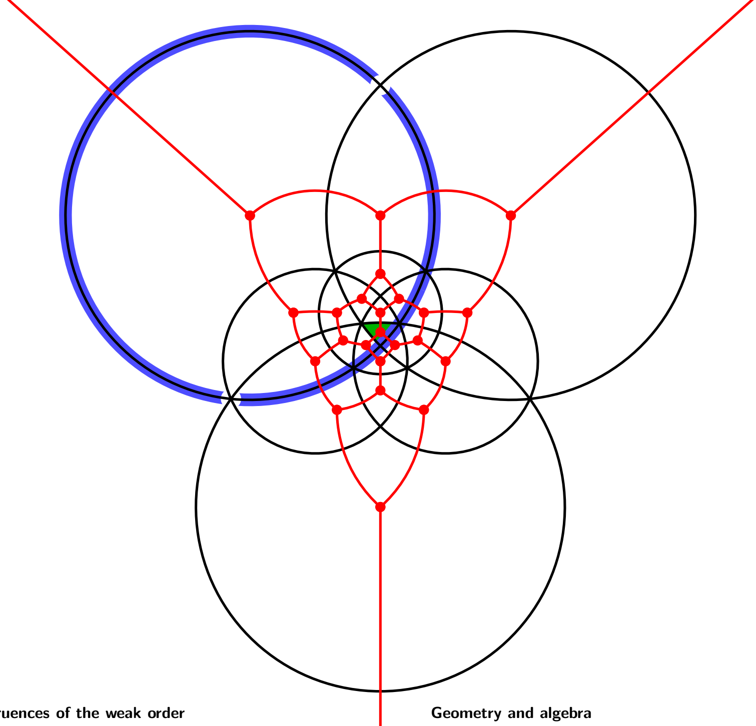


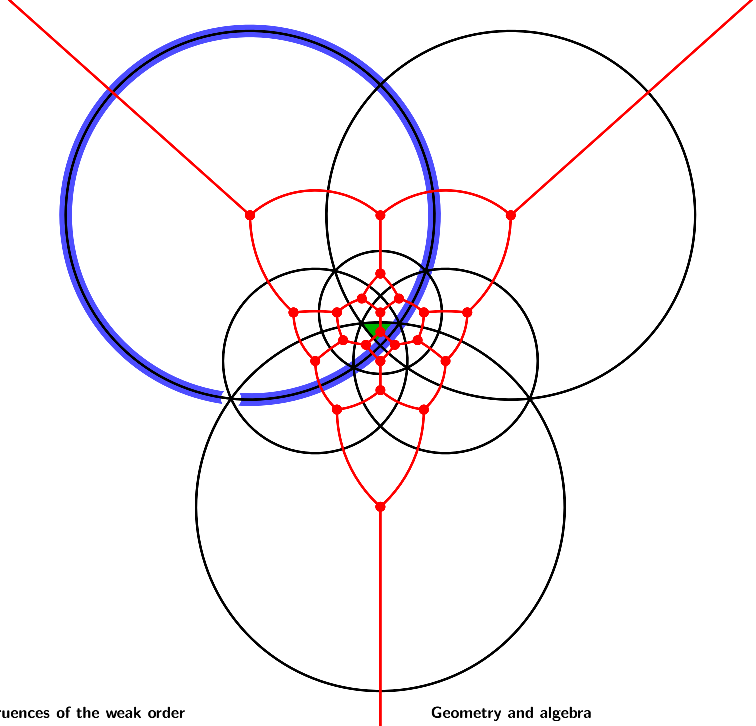


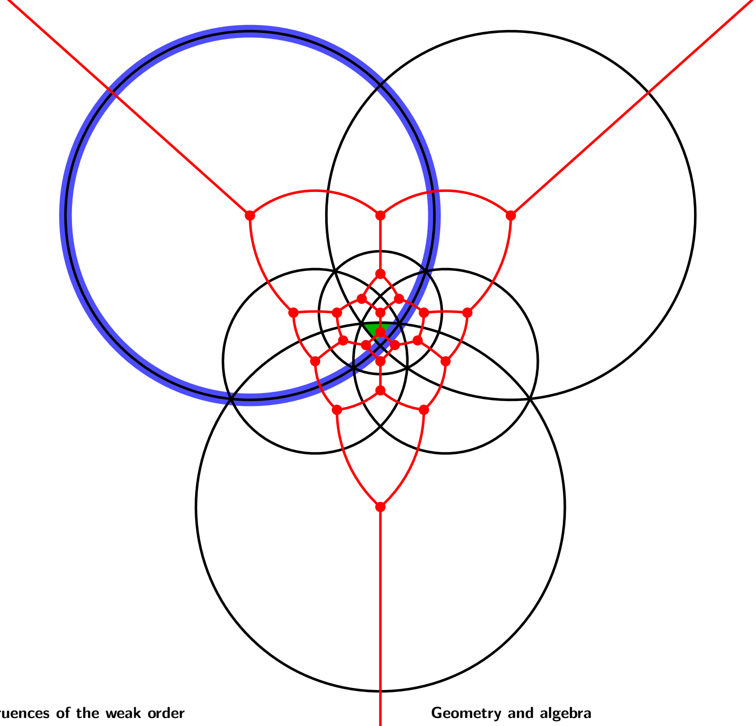


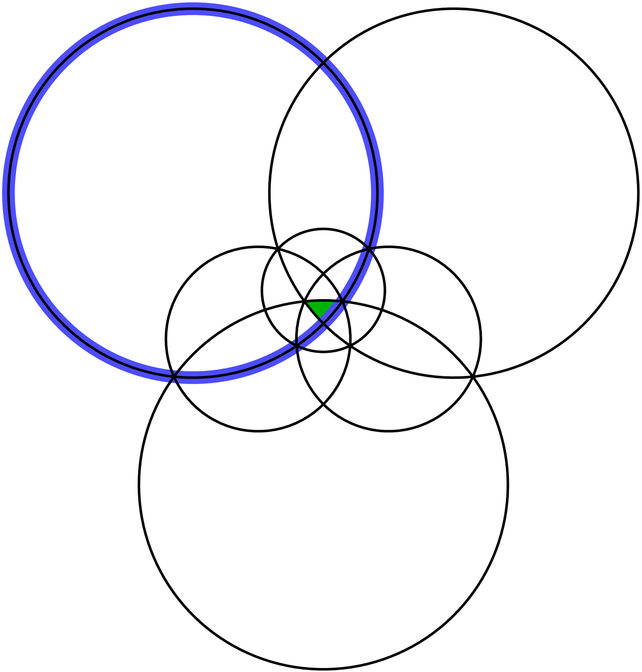


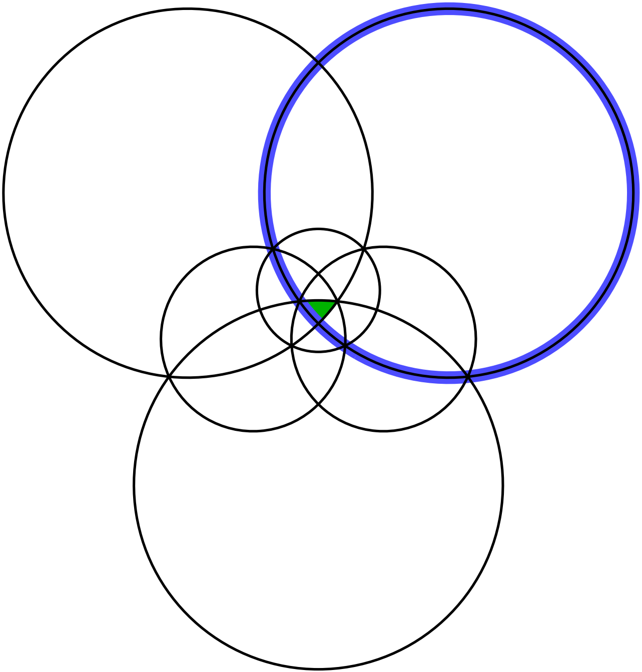


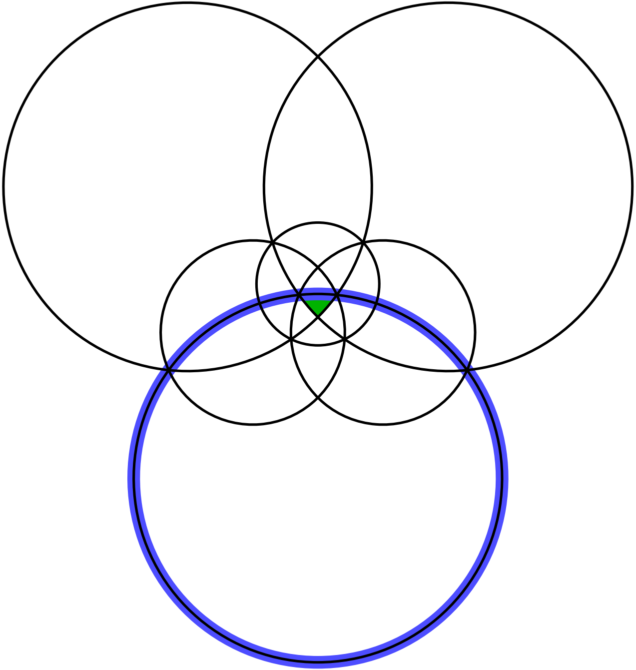


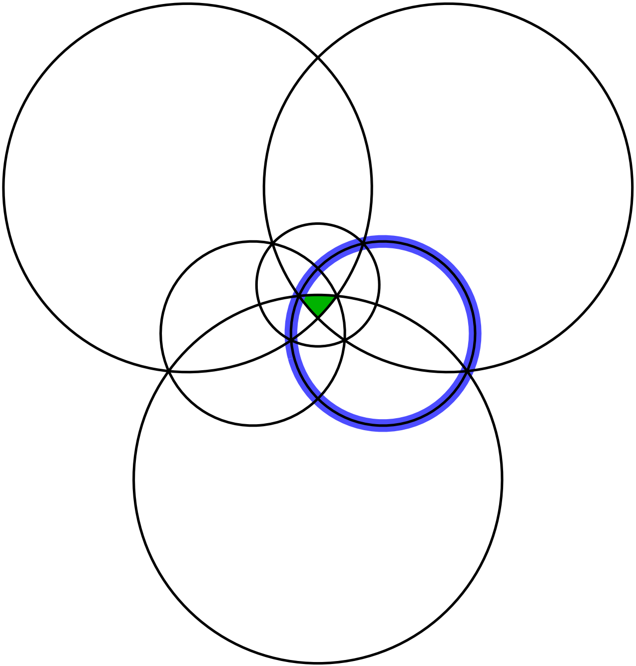


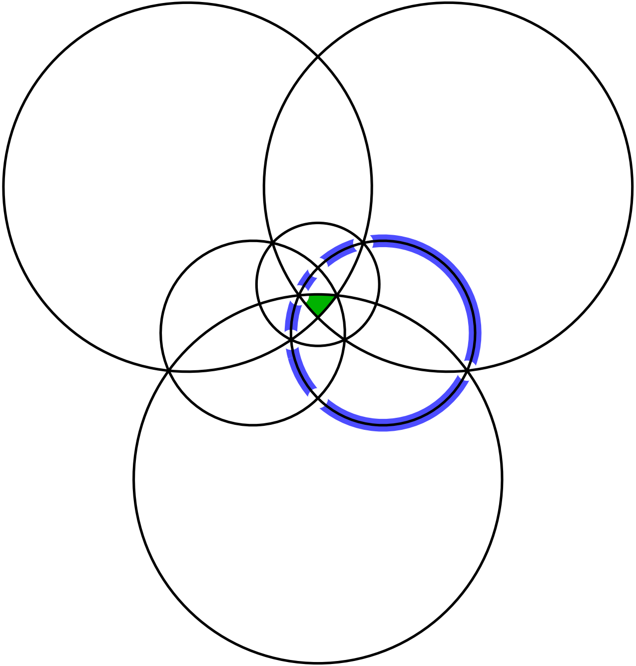


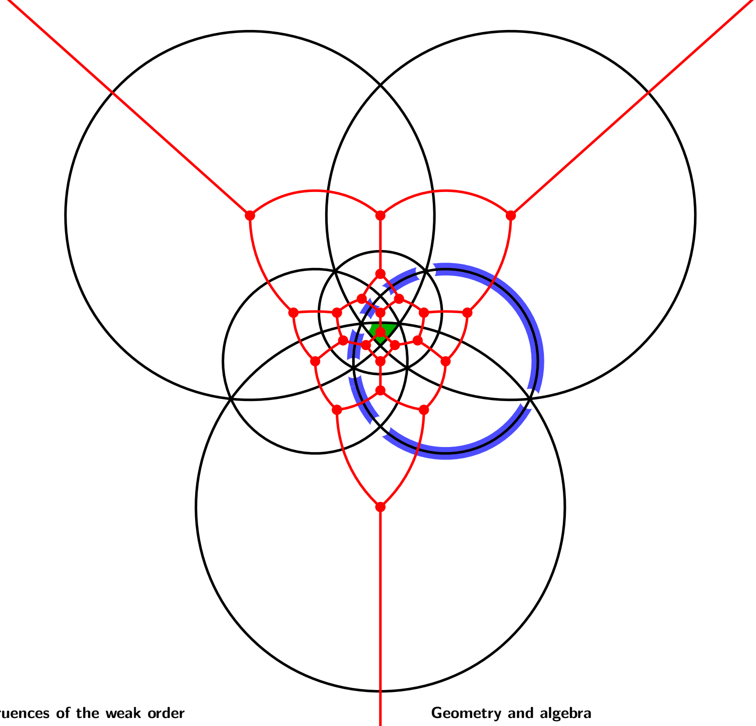


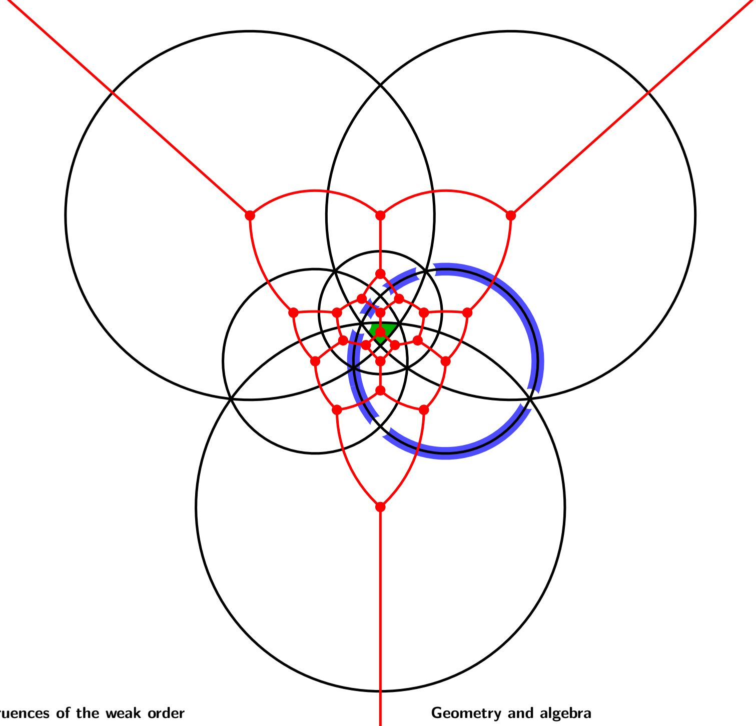


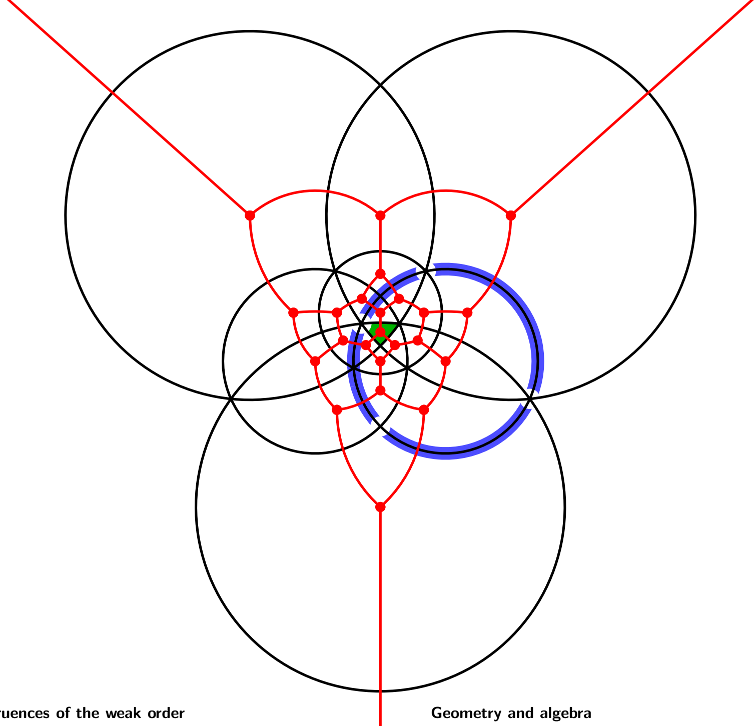


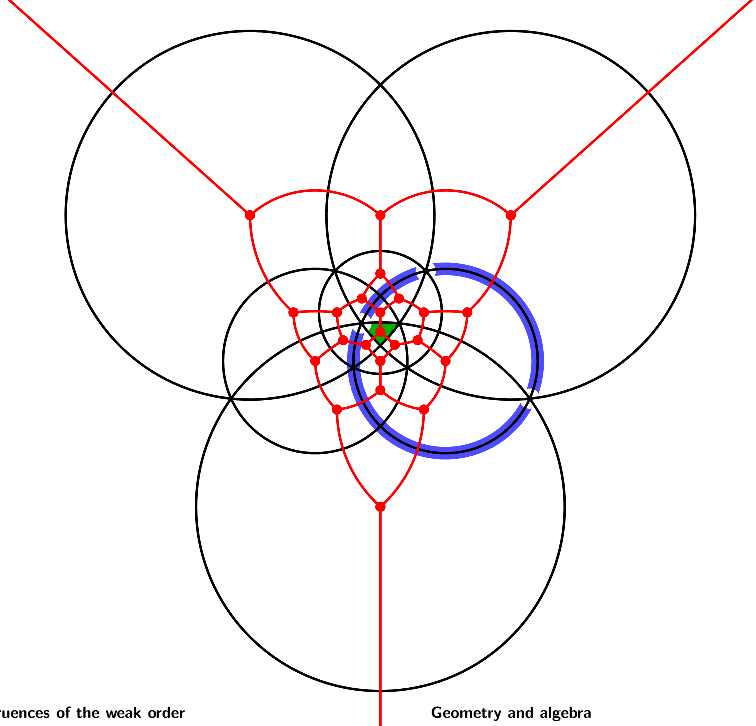


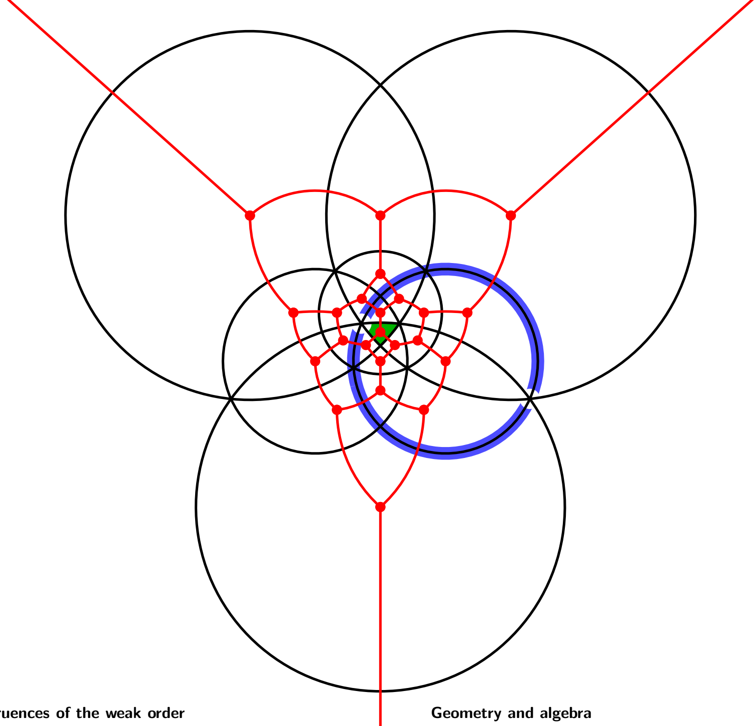


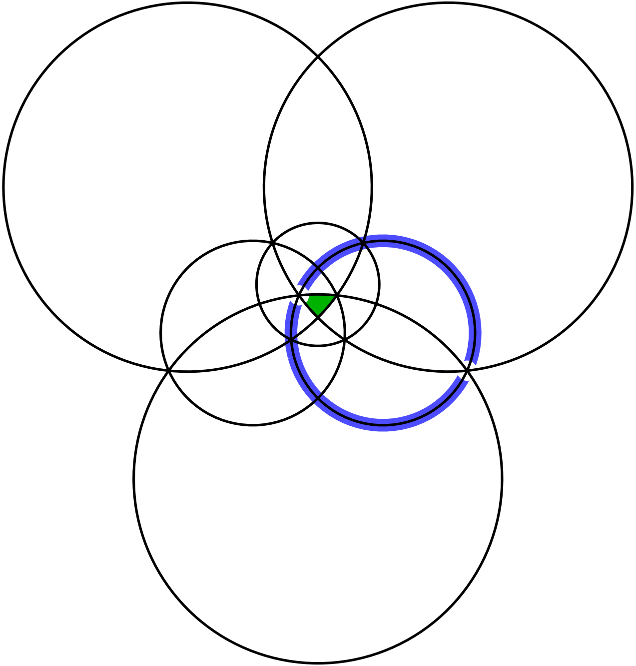


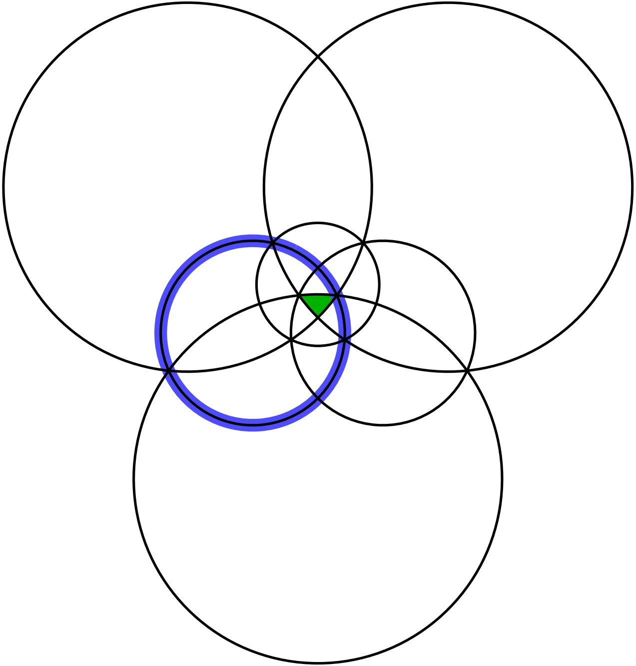


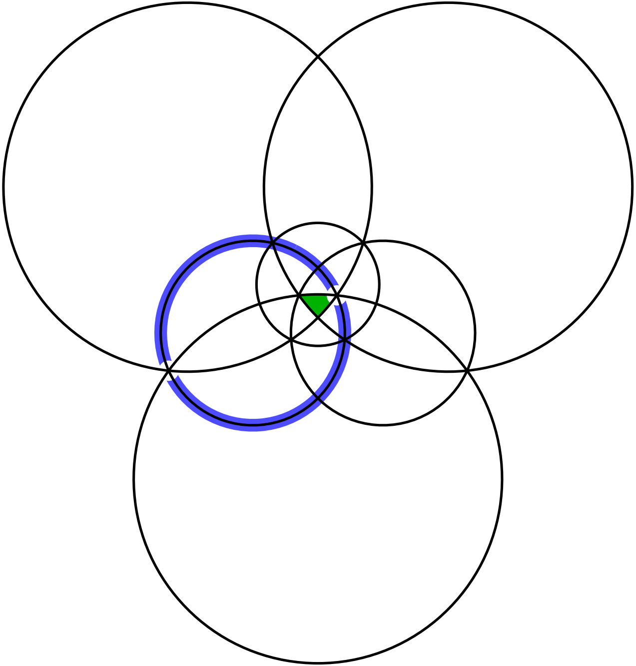


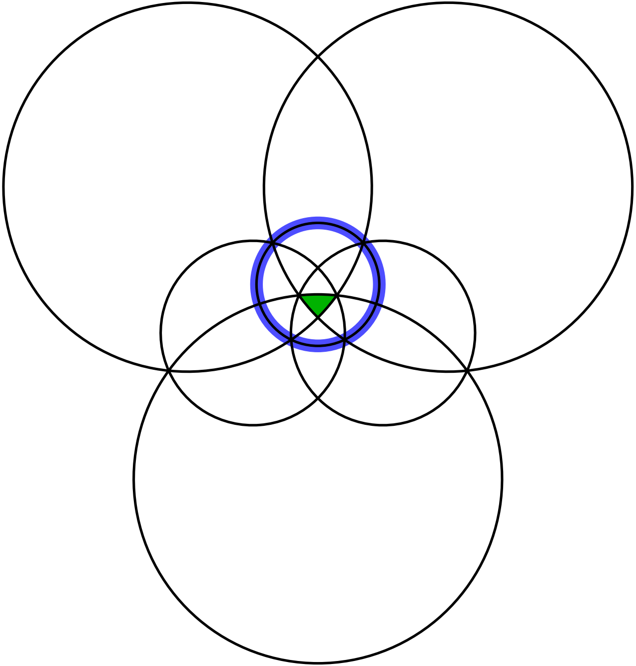


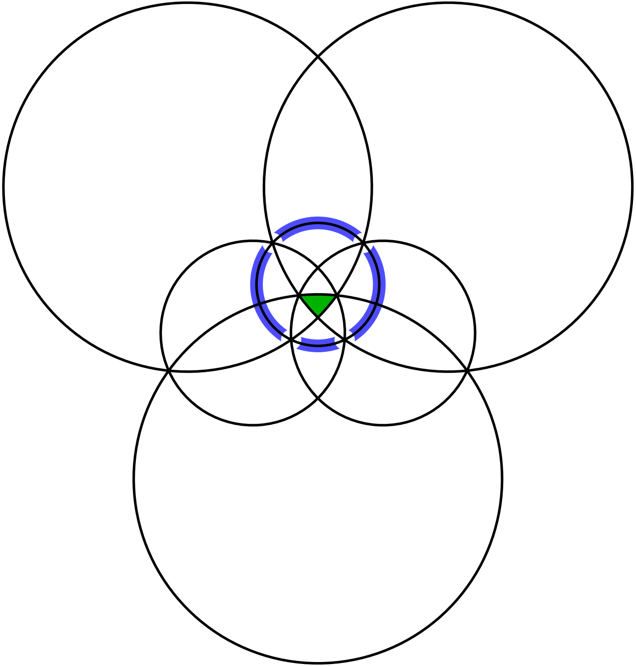


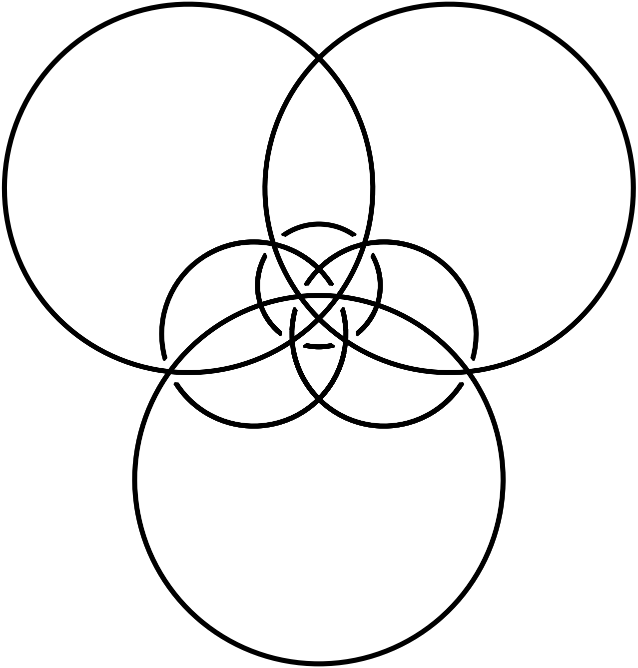




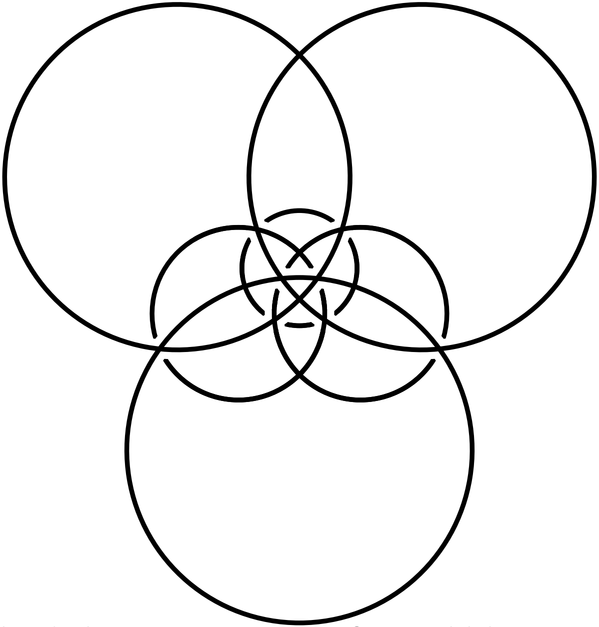




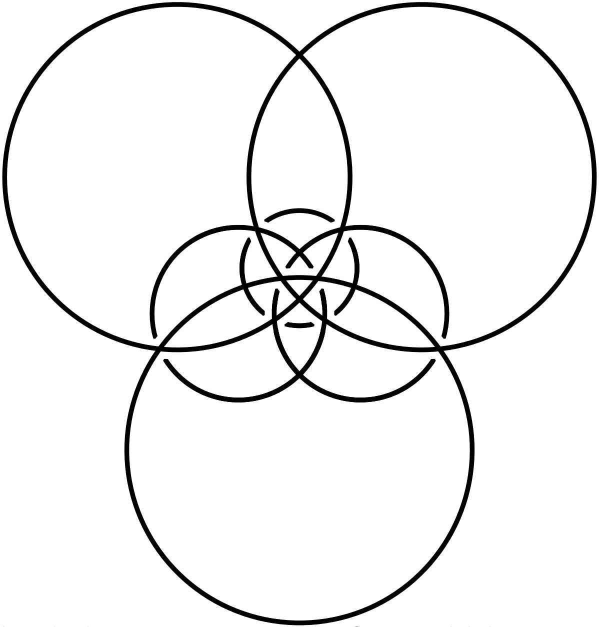




Shards in S_4



Shard removal/forcing example in S_4



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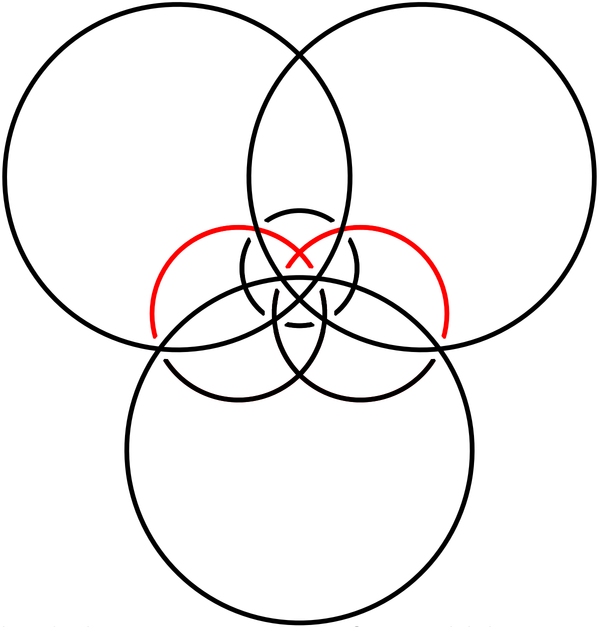

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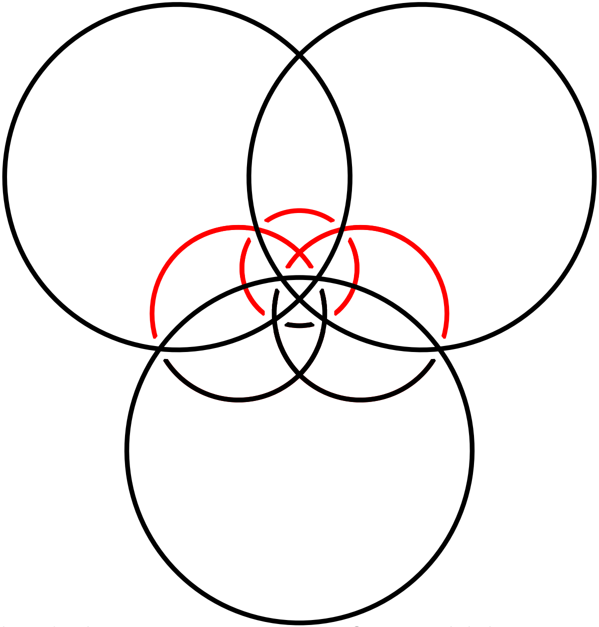

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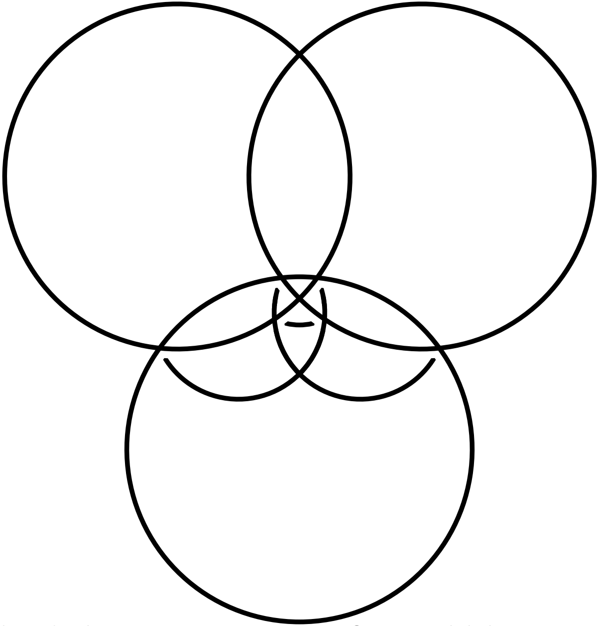
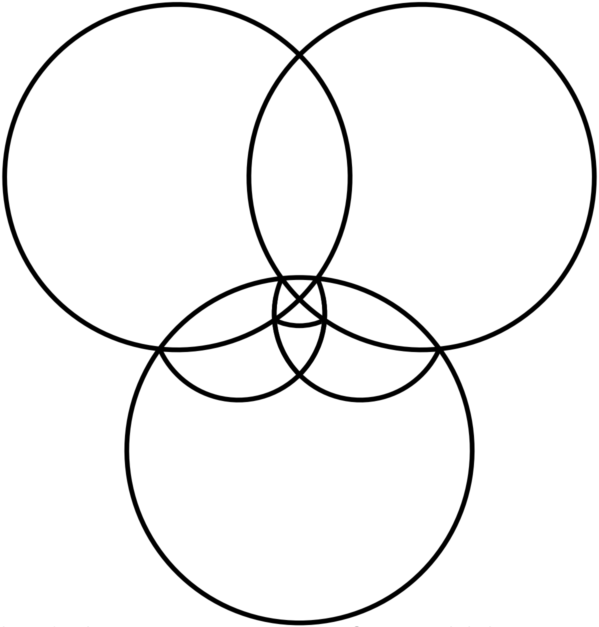


Shard removal/forcing example in S_4 (A Cambrian fan)



Shards in algebra, geometry, and physics

Representation theory of finite dimensional algebras (or quivers):

Theorem (H. Thomas, 2017). The shards associated to W are precisely the closures of domains of semistability of the bricks in the corresponding preprojective algebra.

Mirror symmetry (algebraic geometry, string theory):

Theorem. The shards not removed by the Cambrian congruence constitute the walls of the associated cluster scattering diagram of finite type.

Theorem (R., Stella 2019). For an affine Coxeter group, the shards associated to the doubled Cambrian fan (R., Speyer 2015), and one "shard at infinity" constitute the walls of the associated cluster scattering diagram of affine type.

Future work

For a general (not-necessarily finite) Coxeter group, the weak order is a semilattice, but not a lattice.

With Speyer and Thomas, I am trying to construct a complete lattice associated to any Coxeter group.

In finite type, we know how to build the weak order using the shards: We have found that there is a Fundamental Theorem of Finite Semidistributive Lattices (FTFSDL) generalizing the well-known FTFDL.

One direction of FTFSDL works to create infinite semidistributive lattices, and we think we can use it to make the desired lattice.

The eventual goal is to do everything in complete generality: Congruences, Cambrian congruences, generalized associahedra or their fans, representation theory, cluster scattering diagrams,. . .

References (expository works are highlighted in red)

Barnard and Reading, *Coxeter-biCatalan combinatorics.*

Châtel and Pilaud, *Cambrian Hopf algebras.*

Hohlweg, Lange, Thomas, *Permutahedra and generalized associahedra.*

Law and Reading, *The Hopf algebra of diagonal rectangulations.*

Pilaud, *Brick polytopes, lattice quotients, and Hopf algebras.*

Pilaud and Pons, *Permutrees.*

Pilaud and Santos, *Quotientopes.*

Reading, *From the Tamari lattice to Cambrian lattices and beyond.*

Reading, *Noncrossing arc diagrams and canonical join representations.*

Reading, *Lattice Theory of the Poset of Regions* and *Finite Coxeter Groups and the Weak Order*, Chapters 9–10 in Lattice Theory: STA Volume 2.

Reading, *A combinatorial approach to scattering diagrams.*

Reading and Speyer, *Cambrian fans.*

Reading and Speyer, *Combinatorial frameworks for cluster algebras.*

Reading and Speyer, *Cambrian frameworks for cluster algebras of affine type.*

Reading and Stella, *An affine almost positive roots model.*

Thomas, *Stability, shards, and preprojective algebras.*

Yang, and Zelevinsky, *Cluster algebras of finite type via Coxeter elements...*