k-Schur functions in Catalania

Jennifer Morse

- **Symmetric functions**

- **k-Schur functions**

- **Catalania**
  (joint with Blasiak, Pun, and Summers)

- **Symmetric functions**

- **k-Schur functions**

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Symmetric functions Λ

Multivariate polynomials invariant under S_n-action: $\sigma : x_i \mapsto x_{\sigma(i)}$

$$x_1 + x_2 + x_3$$

$$x_1x_2 + x_1x_3 + x_2x_3$$

$$x_1x_2x_3$$

$$3x_1^2 + 3x_2^2 + 3x_3^2 - 7x_1x_2 - 7x_1x_3 - 7x_2x_3$$

not symmetric:

$$5x_1^2 + 5x_2^2 + 8x_3^2$$

Symmetric functions Λ

Multivariate polynomials invariant under S_n-action: $\sigma : x_i \mapsto x_{\sigma(i)}$

$$e_1 = x_1 + x_2 + x_3$$
$$e_2 = x_1x_2 + x_1x_3 + x_2x_3$$
$$e_3 = x_1x_2x_3$$

Generators

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \qquad \text{or} \qquad h_r = \sum_{i_1\le i_2\le\cdots\le i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$$

symmetric functions are polynomials in the $e_1, e_2, \ldots,$ or in the $h_1, h_2, \ldots$

$$3h_1^2h_2 - h_2^2 + 6h_3h_1 = 3h_{(112)} - h_{(22)} + 6h_{(31)}$$

Schur functions

Raising operator

on functions indexed by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell$:

$$R_{ij} f_\alpha = f_{\alpha+\epsilon_i-\epsilon_j}$$

$$R_{12} h_{(2,2)} = h_{(2+1,2-1)} = h_{(3,1)}$$

$$R_{24} h_{(1,6,2,7,5,1)} = h_{(1,6+1,2,7-1,5,1)}$$

Schur functions

Raising operator

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$$R_{ij} f_\alpha = f_{\alpha+\epsilon_i-\epsilon_j}$$

Schur functions

$$s_\alpha = \prod_{i<j} (1 - R_{ij}) h_\alpha \,,$$

$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

Schur functions

Raising operator

on functions indexed by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell$:

$$R_{ij} f_\alpha = f_{\alpha+\epsilon_i-\epsilon_j}$$

Schur functions

$$s_\alpha = \prod_{i<j} (1 - R_{ij}) h_\alpha \ ,$$

$$s_{211} = (1-R_{12})(1-R_{23})(1-R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - h_{310} + h_{310} + h_{32-1} + h_{400} - h_{41-1}$$

some terms cancel

Schur functions

Raising operator

on functions indexed by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell$:

$$R_{ij} f_\alpha = f_{\alpha+\epsilon_i-\epsilon_j}$$

Schur functions

$$s_\alpha = \prod_{i<j} (1 - R_{ij}) h_\alpha \,,$$

$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

$$s_{13} = (1 - R_{12}) h_{13} = h_{13} - h_{22} = -s_{22}$$

not linearly independent

$$s_{23} = (1 - R_{23}) h_{23} = h_{23} - h_{32} = 0$$

Schur function basis

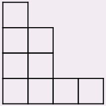
Schur function straightening

$$s_\alpha = \prod_{i<j}(1 - R_{ij})h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 \end{cases}$$

Partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell > 0)$

$$\lambda = (4, 2, 2, 1) = $$

1
2
2
4



- orthonormal basis for Λ
- irreducible S_n-modules
- representatives for Schubert classes in $H^*(Gr)$

Why?

Harmonic polynomials

$\mathcal{M}$ = polynomials killed by all symmetric differential operators
= linear span of all partial derivatives of Vandermonde

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1^1 & 1 \\ x_2^2 & x_2^1 & 1 \\ x_3^2 & x_3^1 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

$\mathcal{M}$ is an S_n-module

$$\mathcal{M} = \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2,\ 2x_2(x_3 - x_1) - x_3^2 + x_1^2,\ x_3 - x_1,\ x_2 - x_3,\ 1\}$$

$1 \mapsto 2 \quad 2 \mapsto 3 \quad 3 \mapsto 1$

Harmonic polynomials

$\mathcal{M}$ = linear span of all partial derivatives of Vandermonde

decompose into irreducible submodules (indexed by partitions)

$$\underbrace{\mathrm{sp}\{\Delta\}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}} \oplus \underbrace{\mathrm{sp}\{1\}}$$

- How many times does a particular submodule occur?

Harmonic polynomials

$\mathcal{M}$ = linear span of all partial derivatives of Vandermonde

decompose into irreducible submodules (indexed by partitions)

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

- How many times does a particular submodule occur?
- Expand $(x_1+\cdots+x_n)^n$ into Schur functions

[Frobenius]

S_n-modules $\longrightarrow$ symmetric functions

harmonic module

irreducible $\mapsto$ s_λ

$\downarrow$

$$(x_1+x_2+x_3)^3 = s_{\ydiagram{1,1,1}} + s_{\ydiagram{1,2}} + s_{\ydiagram{1,2}} + s_{\ydiagram{2}}$$

Combinatorial Bonanza

$$s_{(2,1)} = h_{21} - h_{30}$$

$$= 2\,x_1x_2x_3 + x_1x_1x_2 + x_1x_2x_2 + x_1x_1x_3 + x_1x_3x_3 + x_2x_3x_3 + x_2x_2x_3$$

Combinatorial Bonanza

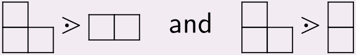
s $= h_{21} - h_{30}$

$$= 2\,x_1x_2x_3 + x_1x_1x_2 + x_1x_2x_2 + x_1x_1x_3 + x_1x_3x_3 + x_2x_3x_3 + x_2x_2x_3$$

Young's Poset

partitions ordered by containment

$\lambda \gtrdot \mu$ when $\mu = \lambda$ - corner box



Combinatorial Bonanza

$s_{21} = h_{21} - h_{30}$

$= 2\,x_1 x_2 x_3 + x_1x_1x_2 + x_1x_2x_2 + x_1x_1x_3 + x_1x_3x_3 + x_2x_3x_3 + x_2x_2x_3$

Young's Poset

partitions ordered by containment

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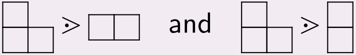
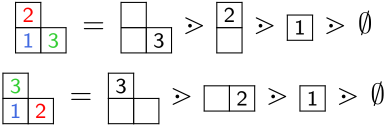


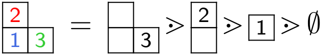
tableaux = saturated chains

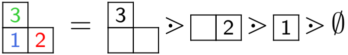
record each time a box is deleted



Tableaux = Pieri rule

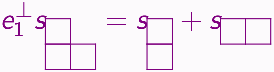
Schur functions = tableaux = saturated chains

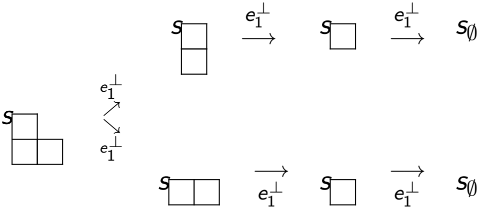




Pieri rule

$$e_1^{\perp} s_\lambda = \sum_{\lambda \gtrdot \mu} s_\mu$$



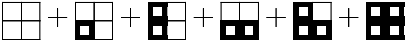


The quantum craze

- Symmetric functions over $\mathbb{Q}(q)$

$$\frac{2}{(1-q)}x_1^2 + \frac{2}{(1-q)}x_2^2 + \frac{17q}{2}x_1x_2 = \frac{2}{(1-q)}s_{\substack{\square\\\square}} + \frac{17q}{2}s_{\square\square}$$

- Representation theory: graded modules
- Geometry: quantum cohomology, string theory
- Combinatorics: q-counting

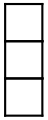


$$\rightarrow (1 + q + 2q^2 + q^3 + q^4)$$

Harmonic polynomials

$\mathcal{M}$ = linear span of all partial derivatives of Vandermonde

$$\underbrace{\mathrm{sp}\{\Delta\}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}} \oplus \underbrace{\mathrm{sp}\{1\}}$$



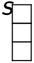
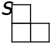
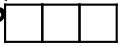

degree 2 polynomials

degree 1 polynomials

How many times does a particular submodule occur?

[Frobenius] harmonic module $\mapsto$ $(x_1+x_2+x_3)^3$

irreducible $\mapsto$ s_λ

$(x_1+x_2+x_3)^3 \quad = \quad s$  $+ \; s$  $+ \; s$  $+ \; s$ 

Harmonic polynomials

$\mathcal{M}$ = linear span of all partial derivatives of Vandermonde

$$\underbrace{\mathrm{sp}\{\Delta\}}_{(1,1,1)} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{(2,1)} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{(2,1)} \oplus \underbrace{\mathrm{sp}\{1\}}_{(3)}$$

degree 2 polynomials

degree 1 polynomials

How many times does a particular submodule occur?

[Frobenius] harmonic module $\mapsto$ $(x_1+x_2+x_3)^3$

irreducible of degree d $\mapsto$ $q^d\, s_\lambda$

$$????? = q^3\, s_{(1,1,1)} + q^2\, s_{(2,1)} + q\, s_{(2,1)} + s_{(3)}$$

Macdonald goes wild (1980's)

Eigenfunctions of: $$\sum_{\substack{I\subset[1,n]\\|I|=1}} \prod_{\substack{i\in I\\j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i} - \sum_{i=1}^{n} t^{-1}$$

where $T_{q,x_i}(f(x_1,\ldots,x_n)) = f(x_1,\ldots,qx_i,\ldots,x_n)$

$$\frac{-4q}{t-1}x_1^2x_2 + \frac{-4q}{t-1}x_2^2x_3 + \frac{-4q}{t-1}x_1^2x_3 + (t^2-7q)x_1x_2x_3$$

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Eigenfunctions of: $$\sum_{\substack{I\subset[1,n]\\|I|=1}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i} - \sum_{i=1}^{n} t^{-1}$$

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[Macdonald:1981] symmetric function basis over $\mathbb{Q}(q,t)$

Macdonald goes wild (1980's)

Eigenfunctions of: $$\sum_{\substack{I\subset[1,n]\\|I|=1}}\prod_{\substack{i\in I\\j\notin I}}\frac{tx_i-x_j}{x_i-x_j}\prod_{i\in I}T_{q,x_i}-\sum_{i=1}^{n}t^{-1}$$

where $T_{q,x_i}(f(x_1,\ldots,x_n))=f(x_1,\ldots,qx_i,\ldots,x_n)$

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[Macdonald:1981] symmetric function basis over $\mathbb{Q}(q,t)$

Conjecture: q,t-positive 'funny basis' expansion

$$J_{2,2}=t^2f_{(4)}+(qt^2+qt+t)f_{(3,1)}+(q^2t^2+1)f_{(2,2)}+(q^2t+qt+q)f_{(2,1,1)}+q^2f_{(1,1,1,1)}$$

q, t-Kostka coefficients

Macdonald polynomials

basis defined (obscurely) as eigenfunctions

Open problem: q, t-enumeration

$\#$ of monomial terms $=$ $\#$ of tableaux

$$J_{2,2} = t^2 f_{\square\square\square\square} + (qt^2 + qt + t) f_{\substack{\square \\ \square\square\square}} + (q^2t^2 + 1) f_{\substack{\square\square \\ \square\square}} + (q^2t + qt + q) f_{\substack{\square \\ \square \\ \square\square}} + q^2 f_{\substack{\square \\ \square \\ \square \\ \square}}$$

[1 2 3 4]   [4 / 1 2 3]   [3 / 1 2 4]   [2 / 1 3 4]   [3 4 / 1 2]   [2 4 / 1 3]   [4 / 3 / 1 2]   [4 / 2 / 1 3]   [3 / 2 / 1 4]   [4 / 3 / 2 / 1]

Garsia-Haiman modules, $\mathcal{M}_\mu$

S_n-module in $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ under $\sigma : x_i y_j \mapsto x_{\sigma(i)} y_{\sigma(j)}$

- $\mathcal{M}_\mu$ = span of partial derivatives of Δ_μ

$$\Delta_{(2,1)} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$\mathcal{M}_{2,1} = \underbrace{\mathrm{sp}\{1\}}_{(0,0):(3)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{(0,1):(2,1)} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{(1,0):(2,1)} \oplus \underbrace{\mathrm{sp}\{\Delta_{(2,1)}\}}_{(1,1):(1,1,1)}$$

$$\tilde{H}_{2,1} = t^0 q^0 s_{(3)} + t^0 q^1 s_{(2,1)} + t^1 q^0 s_{(2,1)} + t^1 q^1 s_{(1,1,1)}$$

- **Symmetric functions**
- ***k*-Schur functions**
- **Catalania**

k-Schur functions in Catalania

by Jennifer Morse

- Symmetric functions
- k-Schur functions
- Catalania

q, t-Kostka coefficients

```
|1|2|3|4| |2|       |3|       |4|       |2|4| |3|4| |3|     |4|     |4|     |4|
          |1|3|4|   |1|2|4|   |1|2|3|   |1|3| |1|2| |2|     |2|     |3|     |3|
                                                    |1|4|   |1|3|   |1|2|   |2|
                                                                             |1|
```

$$H_{(1,1,1,1)} = t^6 s_{(4)} + \left(t^3 + t^4 + t^5\right) s_{(3,1)} + \left(t^2 + t^4\right) s_{(2,2)} + \left(t + t^2 + t^3\right) s_{(2,1,1)} + s_{(1,1,1,1)}$$

$$H_{(2,1,1)} = t^3 s_{(4)} + \left(t + t^2 + qt^3\right) s_{(3,1)} + \left(t + qt^2\right) s_{(2,2)} + \left(1 + qt + qt^2\right) s_{(2,1,1)} + q s_{(1,1,1,1)}$$

$$H_{(2,2)} = t^2 s_{(4)} + \left(qt^2 + qt + t\right) s_{(3,1)} + \left(q^2 t^2 + 1\right) s_{(2,2)} + \left(q^2 t + qt + q\right) s_{(2,1,1)} + q^2 s_{(1,1,1,1)}$$

$$H_{(3,1)} = t s_{(4)} + \left(q^2 t + qt + 1\right) s_{(3,1)} + \left(q^2 t + q\right) s_{(2,2)} + \left(q^3 t + q^2 + q\right) s_{(2,1,1)} + q^3 s_{(1,1,1,1)}$$

$$H_{(4)} = s_{(4)} + \left(q^3 + q^2 + q\right) s_{(3,1)} + \left(q^2 + q^4\right) s_{(2,2)} + \left(q^5 + q^4 + q^3\right) s_{(2,1,1)} + q^6 s_{(1,1,1,1)}$$

q, t-Kostka coefficients

```
|1|2|3|4|  |2|        |3|        |4|        |2|4|  |3|4|  |3|    |4|    |4|    |4|
           |1|3|4|    |1|2|4|    |1|2|3|    |1|3|  |1|2|  |2|    |2|    |3|    |3|
                                                          |1|4|  |1|3|  |1|2|  |2|
                                                                                 |1|
```

$$H_{(1,1,1,1)} = t^6 s_{(4)} + (t^3 + t^4 + t^5)\, s_{(3,1)} + (t^2 + t^4)\, s_{(2,2)} + (t + t^2 + t^3)\, s_{(2,1,1)} + s_{(1,1,1,1)}$$

$$H_{(2,1,1)} = t^3 s_{(4)} + (t + t^2 + qt^3)\, s_{(3,1)} + (t + qt^2)\, s_{(2,2)} + (1 + qt + qt^2)\, s_{(2,1,1)} + q s_{(1,1,1,1)}$$

$$H_{(2,2)} = t^2 s_{(4)} + (qt^2 + qt + t)\, s_{(3,1)} + (q^2 t^2 + 1)\, s_{(2,2)} + (q^2 t + qt + q)\, s_{(2,1,1)} + q^2 s_{(1,1,1,1)}$$

$$H_{(3,1)} = t s_{(4)} + (q^2 t + qt + 1)\, s_{(3,1)} + (q^2 t + q)\, s_{(2,2)} + (q^3 t + q^2 + q)\, s_{(2,1,1)} + q^3 s_{(1,1,1,1)}$$

$$H_{(4)} = s_{(4)} + (q^3 + q^2 + q)\, s_{(3,1)} + (q^2 + q^4)\, s_{(2,2)} + (q^5 + q^4 + q^3)\, s_{(2,1,1)} + q^6 s_{(1,1,1,1)}$$

q, t-Kostka coefficients

2-bounded shapes

$(2,2) \quad (2,1,1) \quad (1,1,1,1)$

$$H_{(1,1,1,1)} = t^4\left(s_{(2,2)} + t s_{(3,1)} + t^2 s_{(4)}\right) + \overbrace{(t^2 + t^3)}^{\text{positive sum of } q,\, t\text{-monomials}} \overbrace{\left(s_{(2,1,1)} + t s_{(3,1)}\right)}^{t\text{-positive sum of Schur functions}} + \left(s_{(1,1,1,1)} + t s_{(2,1,1)} + t^2 s_{(2,2)}\right)$$

$$H_{(2,1,1)} = t\left(s_{(2,2)} + t s_{(3,1)} + t^2 s_{(4)}\right) + (1 + qt^2)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + q\left(s_{(1,1,1,1)} + t s_{(2,1,1)} + t^2 s_{(2,2)}\right)$$

$$H_{(2,2)} = \left(s_{(2,2)} + t s_{(3,1)} + t^2 s_{(4)}\right) + (tq + q)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + q^2\left(s_{(1,1,1,1)} + t s_{(2,1,1)} + t^2 s_{(2,2)}\right)$$

$$H_{(3,1)} = \left(\boxed{q}\, s_{(2,2)} + s_{(3,1)} + t s_{(4)}\right) + (q + q^2)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + q^2\left(q s_{(1,1,1,1)} + qt s_{(2,1,1)} + t s_{(2,2)}\right)$$

$$H_{(4)} = \left(q^2 s_{(2,2)} + q s_{(3,1)} + s_{(4)}\right) + (q^2 + q^3)\left(q s_{(2,1,1)} + s_{(3,1)}\right) + q^4\left(q^2 s_{(1,1,1,1)} + q s_{(2,1,1)} + s_{(2,2)}\right)$$

Bounded Macdonald polynomials

3-bounded shapes

$(3,1)\quad (2,2)\quad (2,1,1)\quad (1,1,1,1)$

positive sum of q, t-monomials · t-positive sum of Schurs · t-positive sum of Schurs

$$H_{(1,1,1,1)} = t^4\big((s_{(2,2)}) + t(s_{(3,1)} + t s_{(4)})\big) + \underbrace{(t^2 + t^3)}_{\text{positive sum of } q,t\text{-monomials}}\underbrace{(s_{(2,1,1)} + t s_{(3,1)})}_{t\text{-positive sum of Schurs}} + \Big(\underbrace{(s_{(1,1,1,1)} + t s_{(2,1,1)})}_{t\text{-positive sum of Schurs}} + t^2 (s_{(2,2)})\Big)$$

$$H_{(2,1,1)} = t\big((s_{(2,2)}) + t(s_{(3,1)} + t s_{(4)})\big) + (1 + qt^2)\big(s_{(2,1,1)} + t s_{(3,1)}\big) + q\Big(\big(s_{(1,1,1,1)} + t s_{(2,1,1)}\big) + t^2 s_{(2,2)}\Big)$$

$$H_{(2,2)} = \big((s_{(2,2)}) + t(s_{(3,1)} + t s_{(4)})\big) + (tq + q)\big(s_{(2,1,1)} + t s_{(3,1)}\big) + q^2\Big(\big(s_{(1,1,1,1)} + t s_{(2,1,1)}\big) + t^2 s_{(2,2)}\Big)$$

$$H_{(3,1)} = \big(q(s_{(2,2)}) + (s_{(3,1)} + t s_{(4)})\big) + (q + q^2)\big(s_{(2,1,1)} + t s_{(3,1)}\big) + q^2\Big(q\big(s_{(1,1,1,1)} + t s_{(2,1,1)}\big) + t s_{(2,2)}\Big)$$

$$H_{(4)} = (q^2 s_{(2,2)} + q s_{(3,1)} + s_{(4)}) + (q^2 + q^3)\big(q s_{(2,1,1)} + s_{(3,1)}\big) + q^4\big(q^2 s_{(1,1,1,1)} + q s_{(2,1,1)} + s_{(2,2)}\big)$$

Conjecture [Lapointe,Lascoux,M:1998]

$$H_{1111} = t^4\Big(\overbrace{\underbrace{s_{22}}_{s^{(3)}_{22}} + t\,\underbrace{\left(s_{31} + t s_{4}\right)}_{s^{(3)}_{31}}\Big)}^{s^{(2)}_{22}} + (t^2+t^3)\overbrace{\underbrace{\left(s_{211} + t s_{31}\right)}_{s^{(3)}_{211}}}^{s^{(2)}_{211}} + \overbrace{\Big(\underbrace{\left(s_{1111} + t s_{211}\right)}_{s^{(3)}_{1111}} + t^2 \underbrace{\left(s_{22}\right)}_{s^{(3)}_{22}}\Big)}^{s^{(2)}_{1111}}$$

$$H_{211} = t\left(s_{22} + t\left(s_{31} + t s_{4}\right)\right) + (1+qt^2)\left(s_{211} + t s_{31}\right) + q\left(\left(s_{1111} + t s_{211}\right) + t^2 s_{22}\right)$$

$$H_{22} = \left(s_{22} + t\left(s_{31} + t s_{4}\right)\right) + (tq+q)\left(s_{211} + t s_{31}\right) + q^2\left(\left(s_{1111} + t s_{211}\right) + t^2 s_{22}\right)$$

For each $k > 0$, there are symmetric functions $s^{(k)}_\lambda(x;t)$ which

- form a basis for the span of k-bounded Macdonald polynomials
- are a t-positive sum of $\{s^{(k+1)}_\mu\}$ (and of Schur functions)

2 decades pass...

Proposed Definition	basis	symmetric	Schur positivity	branching
[1998:Lapointe,Lascoux,M] Tableaux and katabolism		✓	✓	
[2003:Lapointe,M] Jing vertex operators	✓	✓		
[2008:Lam,Lapointe,M,Shimozono] Bruhat order on type-A affine Weyl group				
[2010:Chen,Haiman] $GL_\ell(\mathbb{C})$-equivariant Euler characteristics (Demazure operator)		✓		
[2012:Assaf,Billey] Quasisymmetric functions				
[2015:Dalal,M] Inverting affine Kostka matrix	✓	✓		

Special case when $t = 1$

Definition [2004:Lapointe,M]

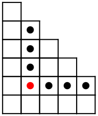
k-Schur functions are functions satisfying Pieri rule
(weak order on type-A affine Weyl group)

k-**Schur functions**	basis	symmetric	positive products	branching
[2004:Lapointe,M] Quantum cohomology of Grassmannian	✓	✓	✓/2	
[2006:Lam] Schubert representatives for the affine Grassmannian	✓	✓	✓	
[2008:Lam,Lapointe,M,Shimozono]	✓	✓	✓	
[2010:Lam,Lapointe,M,Shimozono] Generating functions for marked chains in Bruhat order on type-A affine Weyl group				✓

Underlying combinatorics

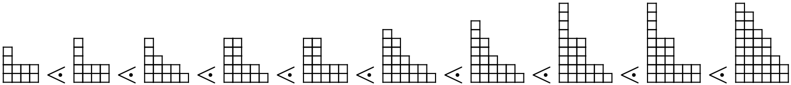
Bruhat order on Grassmannian elements of affine symmetric group

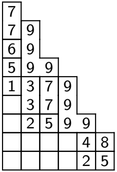
- shapes have no cell with hook-length n



 cell has hook-length 7

- ordered by containment

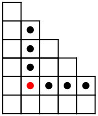




Underlying combinatorics

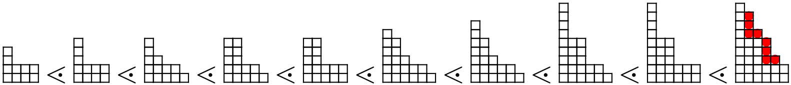
Bruhat order on Grassmannian elements of affine symmetric group

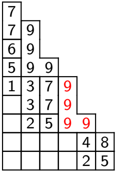
- shapes have no cell with hook-length n



● cell has hook-length 7

- ordered by containment

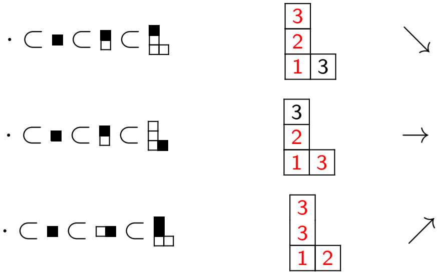




- mark one ribbon in skew of $\lambda \lessdot \mu$

  skew of adjacent shapes = copies of a ribbon

k-Schur functions



$$3\,x_1x_2x_3$$

[Lapointe,Lam,M,Shimozono:2008]

strong marked tableaux generating functions

$$s_\lambda^{(k)} = \sum_T x^{weight(T)}$$

dual Pieri rule given by marked covers in Bruhat order

Theorem at $t = 1$

For each $k > 0$, k-Schur functions are symmetric functions which

- form a basis for $\Lambda^k = \mathbb{Z}[h_1, h_2, \ldots, h_k]$
- are a positive sum of $k+1$-Schur functions
- are a positive sum of Schur functions

structure constants are positive

- generalized Young (Specht) modules
- Gromov-Witten invariants
- Stanley symmetric functions
- WZW conformal field theories
- knot invariants
- affine Stanley functions
- intersections in flag variety
- stable Schubert polynomials
- Hecke algebras at roots of unity
- positroids
- quantum cohomology
- affine Schubert calculus

without direct combinatorial interpretations

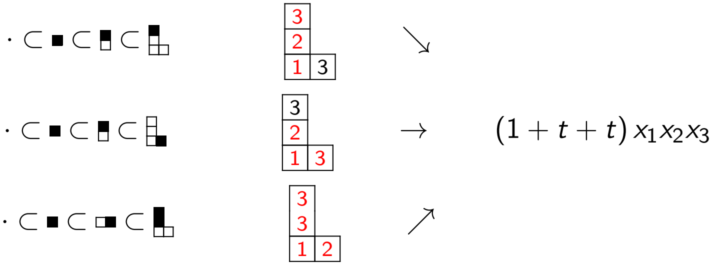
Attempts with the graded case

	basis	symmetric	Schur positivity	branching
[1998:Lapointe,Lascoux,M] Tableaux and katabolism		✓	✓	
[2003:Lapointe,M] Jing vertex operators	✓	✓		
[2008:Lam,Lapointe,M,Shimozono] Bruhat order on type-A affine Weyl group				
[2010:Chen,Haiman] $GL_\ell(\mathbb{C})$-equivariant Euler characteristics (Demazure operator)		✓		
[2012:Assaf,Billey] Quasisymmetric functions				
[2015:Dalal,M] Inverting affine Kostka matrix	✓	✓		

A proposed k-Schur candidate

$$s_\lambda^k(x;t) = \sum_{\text{strong marked tableaux T}} t^{\mathrm{spin}(T)}\, x^T$$

spin encodes # and location of the marked ribbons and their height.



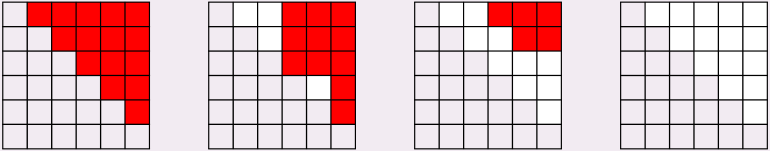
- symmetric?
- basis for bounded Macdonalds?
- Schur and $k+1$-Schur positive?

k-Schur functions in Catalania

- **Symmetric functions**
- **k-Schur functions**
- **Catalania**

Catalan root ideals

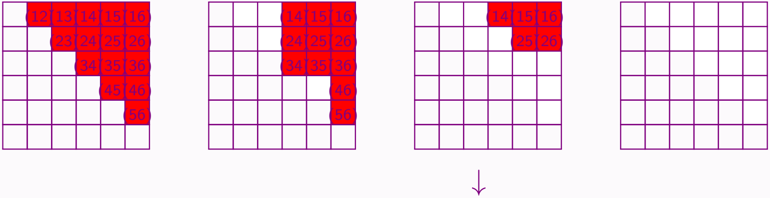
Partitions: flipped, inside a box, above the diagonal



enumerated by Catalan numbers

Catalan root ideals

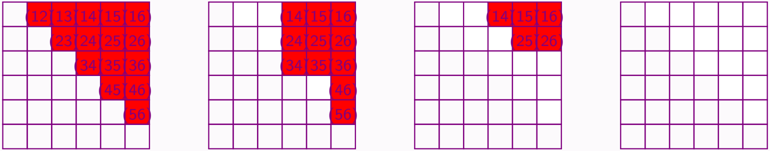
Upper order ideal of roots



$$\Psi = \{(1,4),(1,5),(1,6),(2,5),(2,6)\}$$

Catalan functions

Upper order ideal of roots



$$\downarrow$$

$$\Psi = \{(1,4),(1,5),(1,6),(2,5),(2,6)\}$$

[Panyushev, Chen-Haiman (2010)]

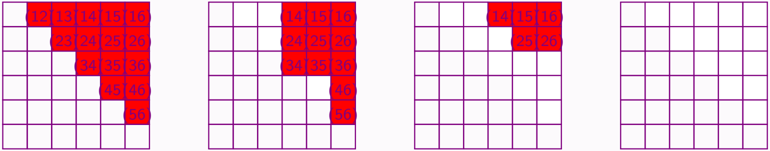
For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x;t) = \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_\gamma(x)$$

G-equivariant Euler characteristics of vector bundles on the flag variety

Catalan functions

Upper order ideal of roots



$\downarrow$

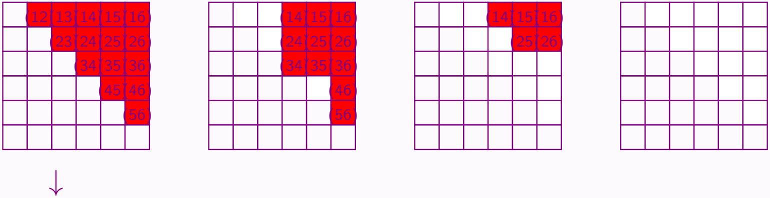
$\Psi = \{\}$

- Schur functions

$$H(\emptyset;\gamma)(x;t) = \prod_{\text{no roots}} (1 - tR_{ij})^{-1} s_\gamma(x) = s_\gamma(x)$$

Catalan functions

Upper order ideal of roots



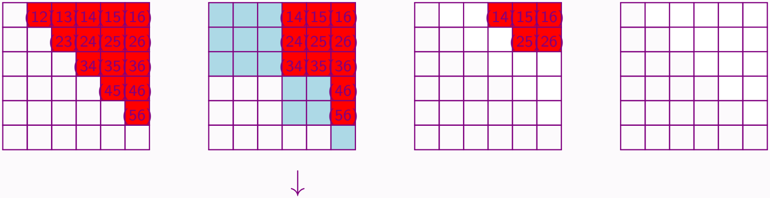
$\Psi = \{\text{all roots}\}$

- Macdonald polynomials at $q = 0$

$$H(\Delta^+; \lambda)(x; t) = \prod_{i<j} (1 - tR_{ij})^{-1} s_\lambda(x)$$

Catalan functions

Upper order ideal of roots



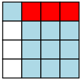
$\Psi = \{$roots lie above blocks$\}$

- Parabolic Hall-Littlewood functions

$$H(\emptyset;\gamma)(x;t) = \prod_{\text{parabolic ideal}} (1 - tR_{ij})^{-1} s_\gamma(x) = s_\gamma(x)$$

Parabolic case

$$H(\Psi; (4322)) = \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_{4322}$$

$\Psi =$  $= \{(1,2),(1,3),(1,4)\}$

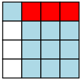
$$= (1+tR_{14}+t^2R_{14}^2+\cdots)(1+tR_{13}+t^2R_{13}^2+\cdots)(1+tR_{12}+t^2R_{12}^2+\cdots)\, s_{4322}$$

$$= (1+tR_{13}+t^2R_{13}^2+t^3R_{13}^3+\cdots+tR_{14}+t^2R_{14}R_{13}+\cdots)\, s_{4322}$$

$$= s_{4322} + ts_{5312} + t^2 s_{6302} + t^3 s_{73-12} + \cdots + ts_{5321} + t^2 s_{6311} + \cdots$$

Parabolic case

$$H(\Psi;(4322)) = \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_{4322}$$

$\Psi =$  $= \{(1,2),(1,3),(1,4)\}$

$$= (1+tR_{14}+t^2R_{14}^2+\cdots)(1+tR_{13}+t^2R_{13}^2+\cdots)(1+tR_{12}+t^2R_{12}^2+\cdots)\, s_{4322}$$

$$= (1+tR_{13}+t^2R_{13}^2+t^3R_{13}^3+\cdots+tR_{14}+t^2R_{14}R_{13}+\cdots)\, s_{4322}$$

$$= s_{4322} + ts_{5312} + t^2 s_{6302} + t^3 s_{73-12} + \cdots + ts_{5321} + t^2 s_{6311} + \cdots$$

straightening introduces negatives!

$$= s_{4322} + 0 - t^2\, s_{6311} - t^3\, s_{7310} + \cdots + t\, s_{5321} + t^2\, s_{6311} + \cdots$$

Catalan functions

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x;t) = \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_\gamma(x)\,,$$

G-equivariant Euler characteristics of vector bundles on the flag variety

Conjecture

When λ is a partition, $H(\Psi;\lambda)(x;t)$ is Schur positive

Conjecture [Chen,Haiman:2010]

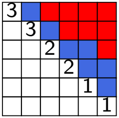
k-Schur functions are some subclass of $H(\Psi;\lambda)(x;t)$

Distinguished Catalan functions

k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i, i+j) : j > k - \lambda_i\}$$

$= \text{root ideal with } k - \lambda_i$ non-roots in row i

$\Delta^4(3,3,2,2,1,1) =$ 

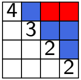
$\leftarrow$ row i has $4 - \lambda_i$ non-roots

k-Schur Catalan function

$$\mathfrak{s}_\lambda^{(k)} = H(\Delta^k(\lambda), \lambda) = \prod_{(i,j)\in\Delta^k(\lambda)} (1 - tR_{ij})^{-1} s_\lambda$$

k-Schur Catalan function

$$\mathfrak{s}^5_{4322} = \prod_{(i,j)\in\Delta^5(4322)} (1 - tR_{ij})^{-1} s_{4322}$$

$\Delta^5(4322) =$ 

$= \{(1,3),(1,4)\}$

$$= (1+tR_{14}+t^2R_{14}^2+\cdots)(1+tR_{13}+t^2R_{13}^2+\cdots)\, s_{4322}$$

$$= (1+tR_{13}+t^2R_{13}^2+t^3R_{13}^3+\cdots+tR_{14}+t^2R_{14}R_{13}+\cdots)\, s_{4322}$$

$$= s_{4322} + ts_{5312} + t^2 s_{6302} + t^3 s_{73-12} + \cdots + ts_{5321} + t^2 s_{6311} + \cdots$$

straightening introduces negatives!

$$= s_{4322} + 0 - t^2\, s_{6311} - t^3\, s_{7310} + \cdots + t\, s_{5321} + t^2\, s_{6311} + \cdots$$

Some things are easier

Pieri rule for Catalan functions

$$e_1^\perp H(\Psi;\gamma) = \sum_i H(\Psi;\gamma - \epsilon_i)$$

Pieri for Schur functions

$$e_1^\perp s_\lambda = \sum_{\lambda \gtrdot \mu} s_\mu \qquad \mu = \lambda - \text{corner box}$$

Pieri for strong tableaux k-Schur functions

$$e_1^\perp s_\lambda^{(k)} = \sum_{\lambda \gtrdot_B \mu} t^{\mathrm{spin}} s_\mu^{(k)} \qquad \mu = \lambda - \text{ribbons/marking}$$

Schur case: $s_\alpha = H(\emptyset; \alpha)$

Pieri rule for Catalan functions

$$e_1^\perp H(\Psi; \gamma) = \sum_i H(\Psi; \gamma - \epsilon_i)$$

$$e_1^\perp H(\emptyset, \ydiagram{2,3}) = H(\emptyset, \ydiagram{2,3}) + H(\emptyset, \ydiagram{2,2}) + H(\emptyset, \ydiagram{1,2,3})$$

Schur straightening $$H(\emptyset : \lambda - e_i) = \begin{cases} s_\mu & \text{for } \mu = \lambda - e_i \lessdot \lambda \\ 0 & \end{cases}$$

Schur Pieri

$$e_1^\perp s_\lambda = \sum_{\mu \lessdot \lambda} s_\mu \qquad \text{e.g. } e_1^\perp s_{\ydiagram{2,2,3}} = s_{\ydiagram{2,2,2}} + s_{\ydiagram{1,2,3}}$$

k-Schur case: $H(\Delta^k(\lambda), \lambda)$

Pieri rule for Catalan functions

$$e_1^{\perp} H(\Psi;\gamma) = \sum_i H(\Psi;\gamma - \epsilon_i)$$

$$e_1^{\perp} H(\Delta^k(322), \ydiagram{2,2,3}) = H(\Delta^k(322), \ydiagram{2,1,3}) + H(\Delta^k(322), \ydiagram{2,2,2}) + H(\Delta^k(322), \ydiagram{1,2,3})$$

↑

straightening won't cut it need $\Delta^k(222)$

k-Schur Pieri

$$e_1^{\perp} H(\Delta^k(\lambda);\lambda) = \sum t^* H(\Delta^k(\mu);\mu)$$

Shocking discovery

Shift invariance

$$e_\ell^\perp \mathfrak{s}^{(k+1)}_{\lambda+1^\ell} = \mathfrak{s}^{(k)}_\lambda$$

Shocking discovery

Shift invariance

$$e_\ell^\perp \mathfrak{s}^{(k+1)}_{\lambda+1^\ell} = \mathfrak{s}^{(k)}_\lambda$$

- mysterious branching

$$\mathfrak{s}^{(k)}_\lambda = \sum ???? \, \mathfrak{s}^{(k+1)}_\nu$$

- given by dual Pieri rule!

$$e_\ell^\perp \mathfrak{s}^{(k+1)}_\gamma = \sum c_{\gamma\nu}(t) \, \mathfrak{s}^{(k+1)}_\nu$$

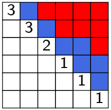
$$\uparrow$$

$$\gamma = \lambda + 1^\ell$$

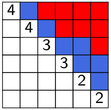
Shift invariance for $H\left(\Delta^k(\lambda); \lambda\right)$

Property: $\Delta^k(\lambda) = \Delta^{k+1}(\lambda + 1^\ell)$

$$
\begin{aligned}
\#\text{ non-roots in row } i \text{ of } \Delta^k(\lambda) &= k - \lambda_i \\
&= (k+1) - (\lambda_i + 1) \\
&= \#\text{ non-roots in row } i \text{ of } \Delta^{k+1}(\lambda + 1^\ell)
\end{aligned}
$$

$\Delta^4(3,3,2,1,1,1) =$ 

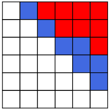
$\leftarrow$ 4 - 2 non-roots

$\Delta^5(4,4,3,3,2,2) =$ 

$\leftarrow$ 5 - 3 non-roots

Shift invariance

Property: $\Delta^k(\lambda) = \Delta^{k+1}(\lambda + 1^\ell)$

$\Delta^4(3,3,2,1,1,1) =$  $= \Delta^5(4,4,3,3,2,2)$

Catalan Pieri

$$e_\ell^\perp H(\Psi; \lambda + 1^\ell) \; = H(\Psi; \lambda)$$

$\Delta^{k+1}(\lambda+1^\ell)=\Delta^k(\lambda)$

Theorem

$$e_\ell^\perp H\left(\Delta^{k+1}(\lambda + 1^\ell); \lambda + 1^\ell\right) = H\left(\Delta^k(\lambda); \lambda\right)$$

Dual Pieri Rules

k-Schur Pieri: $\mathfrak{s}^k_\lambda = H(\Delta^k(\lambda); \lambda)$

$$e_1^\perp \mathfrak{s}^k_\lambda = \sum_{\mu \lessdot_B \lambda} t^{\text{spin}}\, \mathfrak{s}^k_\mu$$

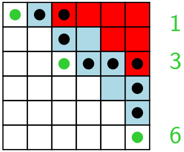
Catalan Pieri $\quad e_1^\perp H(\Psi; \lambda) = \sum_r H(\Psi; \lambda - e_r)$

$$e_1^\perp H(\Delta^k(\lambda); \lambda) = \sum_{i \geq 1} \underbrace{H(\Delta^k(\lambda); \lambda - e_i)}_{\text{not a } k\text{-Schur Catalan}}$$

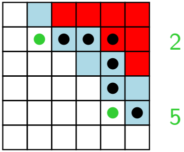
Downpaths

$$H(\Delta^k(\lambda); \lambda - e_i) \quad = \sum_{b \in \text{downpath}(i)} t^{\#\text{ bounces }[i,b]} H(\Delta^k(\lambda - e_b); \lambda - e_b)$$

downpath(1)=



downpath(2)=



$$H(\Delta^5(433332); 433332 - e_2)$$

$$= \quad H(\Delta^5(423332); 423332) + tH(\Delta^5(433322); 433322)$$

Dual Pieri Rules

k-Schur case: $\mathfrak{s}^k_\lambda = H(\Delta^k(\lambda);\lambda)$

$$e_1^\perp \mathfrak{s}^k_\lambda = \sum_{\mu \lessdot_B \lambda} t^{\text{spin}}\, \mathfrak{s}^k_\mu$$

key property $$e_1^\perp H(\Psi;\lambda) = \sum_r H(\Psi;\lambda - e_r)$$

$$e_1^\perp H(\Delta^k(\lambda);\lambda) = \sum_{i \geq 1} \sum_{b \in \text{downpath}(i)} t^{\text{bounce[i,b]}} H(\Delta^k(\lambda - e_b);\lambda - e_b)$$

straightening: $$H(\Delta^k(\lambda - e_b);\lambda - e_b) = \begin{cases} t^h\, H(\Delta^k(\mu);\mu) & \text{for } \mu \lessdot_B \lambda \\ 0 & \end{cases}$$

k-Schur Catalan functions

Dual Pieri rule

$$e_\ell^\perp \mathfrak{s}_{\lambda+1^\ell}^{(k+1)}(x;t) = \sum_{\nu \lessdot \lambda+1^\ell} t^{\mathrm{spin}}\, \mathfrak{s}_\nu^{(k+1)}(x;t)$$

Shift invariance

$$e_\ell^\perp\, \mathfrak{s}_{\lambda+1^\ell}^{(k+1)} = \mathfrak{s}_\lambda^{(k)}$$

Corollaries

- Branching
$$\mathfrak{s}_\lambda^{(k)}(x;t) = \sum_\nu t^{\mathrm{spin}}\, \mathfrak{s}_\nu^{(k+1)}(x;t)$$
- k-Schur Catalans = spin strong tableaux functions

Theorem [Blasiak,M,Pun,Summers]

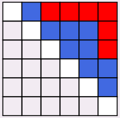
	basis	symmetric	Schur positivity	branching
[1998:Lapointe,Lascoux,M] Tableaux and katabolism		✓	✓	
[2003:Lapointe,M] Jing vertex operators	✓	✓	✓	✓
[2008:Lam,Lapointe,M,Shimozono] Bruhat order on type-A affine Weyl group	✓	✓	✓	✓
[2010:Chen,Haiman] $GL_\ell(\mathbb{C})$-equivariant Euler characteristics (Demazure operator)	✓	✓	✓	✓
[2012:Assaf,Billey] Quasisymmetric functions	✓	✓	✓	✓
[2015:Dalal,M] Inverting affine Kostka matrix	✓	✓		
[2018:Blasiak,M,Pun,Summers] Catalan functions	✓	✓	✓	✓

Future?

- k-Schur function expansion of k-bounded Macdonald polynomials

$$H_{22} = \overbrace{\left(s_{22} + t s_{31} + t^2 s_{4}\right)}^{s^{(2)}_{22}} + (tq+q)\overbrace{\left(s_{211} + t s_{31}\right)}^{s^{(2)}_{211}} + q^2 \overbrace{\left(s_{1111} + t s_{211} + t^2 s_{22}\right)}^{s^{(2)}_{1111}}$$

- Conjecture: Catalan functions are Schur positive for dominant weights
  Refinement: k-Schur positivity when non-roots are bounded by k



maximum non-roots in row is 3

- affine Schubert calculus