Some equivariant combinatorics of flag manifolds

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November 10, 2018

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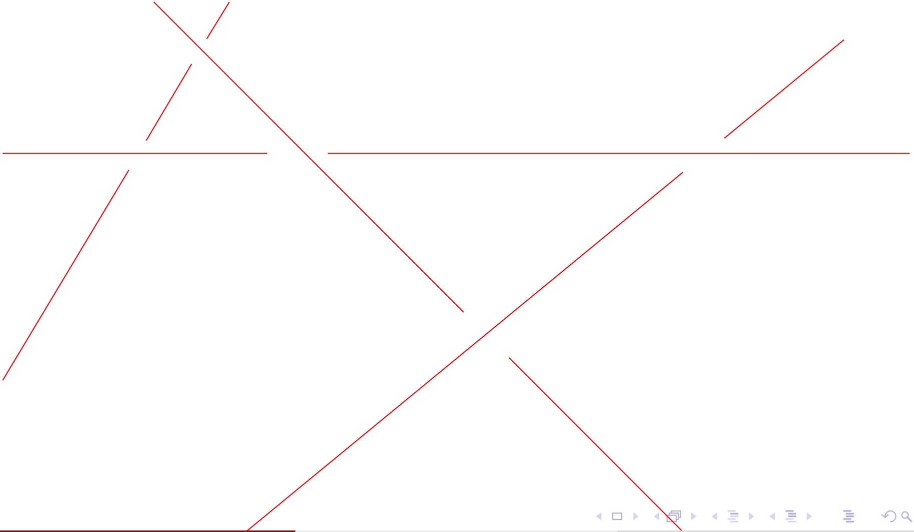

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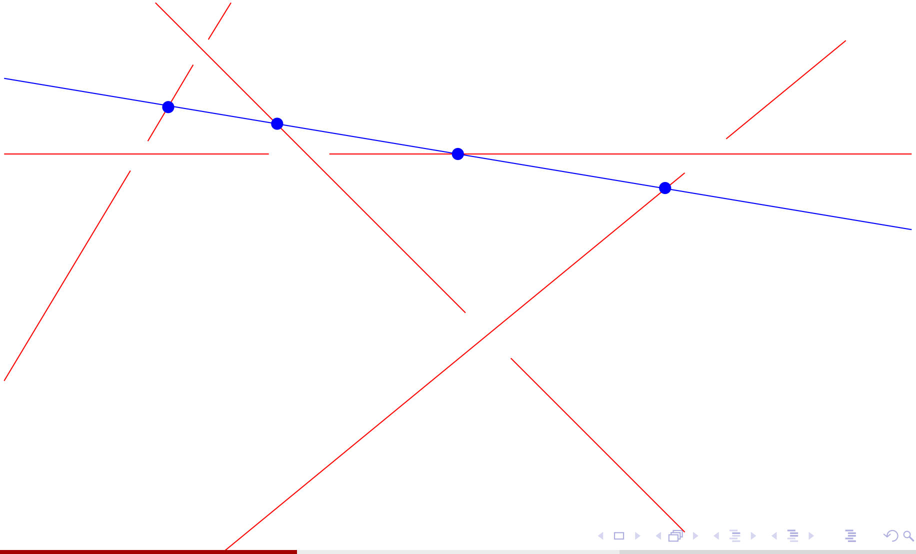

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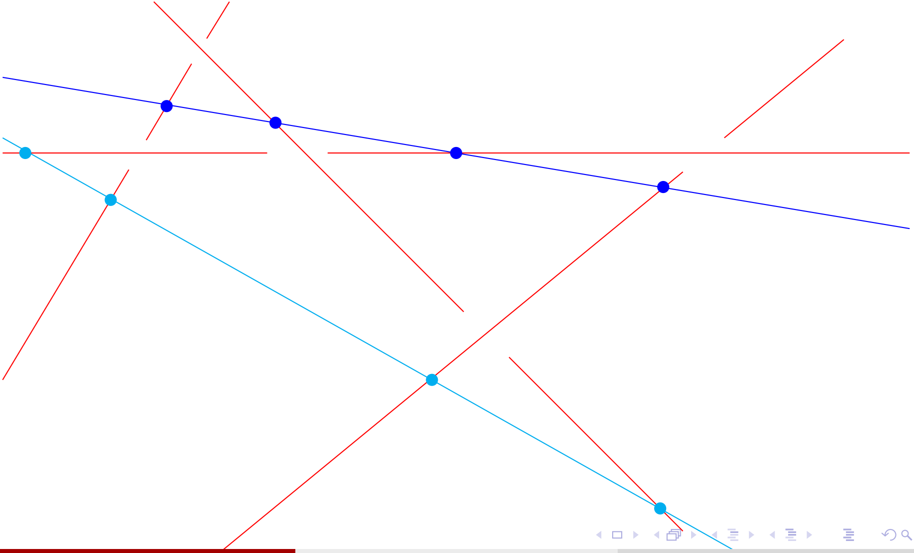

Rephrasing with complex geometry

The question can be phrased as an intersection of algebraic varieties.

- Replace a *line in 3-space* by a *plane through the origin in 4-space*.
- Move to complex planes in $\mathbb{C}^4$.

$$Gr(2,4) = \{\text{2-dimensional vector spaces in } \mathbb{C}^4.\}$$

- Each line we started with is a point in $Gr(2,4)$.
- Let X_i be the set of planes in $\mathbb{C}^4$ that intersect the ith red line. Then $X_i \subset Gr(2,4)$.
- Our question is answered by the number of points in

$$X_1 \cap X_2 \cap X_3 \cap X_4.$$

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Flags (type A)

Definition

A flag on $\mathbb{C}^n$ is a sequence of vector spaces with increasing dimensions:

$$0 \subset V_1 \subset V_2 \subset \cdots \subset \mathbb{C}^n,$$

with $\dim_{\mathbb{C}} V_i = i$. The flag manifold $Fl(n, \mathbb{C})$ is the collection of all flags.

Flags of type A are also realized as $Gl(n, \mathbb{C})/B$, where B consists of upper triangular matrices.

More generally, for G a complex reductive Lie group and B a Borel subgroup, G/B is called a flag variety. Note it has a complex structure coming from G itself. We will also pick $T \subset B$ a maximal torus in B. For $G = Gl(n, \mathbb{C})$ and B upper triangular matrices, T consists of diagonal matrices with non-zero entries along the diagonal.

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Schubert varieties

Let $W := N(T)/T$, the normalizer of the torus mod the torus. This is a finite group (and isomoprhic to S_n for $Gl(n, \mathbb{C})$.) *Schubert varieties* are defined by

$$X^w := \overline{BwB/B},$$

for each $w \in W$. Similarly, there are varieties $X_w := \overline{B_- wB/B}$ where B_- consists of lower triangular matrices (or an opposite Borel).
Schubert calculus is the study of intersection properties some special subvarieties of $Fl(n, \mathbb{C})$. . Specifically,

$$c_{uv}^w = [X_u] \cap [X_v] \cap [X^w]$$

is nonnegative integer if the varieties are the right dimension. So how can we count it?

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A group action on $Fl(n, \mathbb{C})$

Let $\mathbb{C}^n \cong \mathbb{C}_1 \oplus \cdots \oplus \mathbb{C}_n$ be given by a choice of basis. For $T \cong S^1 \times \cdots \times S^1$, consider the action of T on $\mathbb{C}^n$ given by

$$(\theta_1, \cdots, \theta_n) \cdot (z_1, \cdots, z_n) = (e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n).$$

Note that each summand is an eigenspace of the action.

- For V a vector space, $T \cdot V$ is a vector space of the same dimension.
- If $V \subset W$, then $T \cdot V \subset T \cdot W$.
- Therefore, T acts on $Fl(n, \mathbb{C})$.
- The fixed points are coordinate planes. For each permutation σ of the $\{1, \cdots n\}$, the coordinate flag

$$0 \subset \mathbb{C}_{\sigma(1)} \subset \mathbb{C}_{\sigma(1)} \oplus \mathbb{C}_{\sigma(2)} \subset \cdots \subset \mathbb{C}^n$$

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- The fixed points are in 1-1 correspondence with S_n, the permutation group on n letters.

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What can be said about c_{uv}^w?

- The same numbers can be obtained as structure constants for the *basis* of Schubert varieties in the cohomology $H^*(G/B)$. Let S_w represent the cohomology class $[X_w]$.

$$S_u S_v = \sum_{w \in W} c_{uv}^w S_w.$$

- Positive (combinatorial) formula known for these constants in the case of Grassmannians. They are counted by Young tableaux.
- Positive formulas are known for the general flag manifold in special cases, i.e. for a subset of Weyl group elements u.
- The structure constants count the number of times a irreducible representation occurs in the tensor product of two other representations.

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Generalizing the coefficients c_{uv}^w

- In the equivariant cohomology $H_T^*(G/B)$, there is a basis given by cohomology classes representing X_w, denoted by S_w^T. Then

$$S_u^T S_v^T = \sum_{w \in W} (c_{uv}^w)^T S_w^T.$$

defines coefficients $(c_{uv}^w)^T \in H_T^*(pt)$.

- $$H_T^*(pt) = \mathbb{Z}[x_1, \ldots, x_n]$$

- These are also positive in an appropriate sense. With choosing a Borel B, we also choose a set of positive roots. In this case, $\Delta^+ = \{x_i - x_j : i < j\}$. Then $(c_{uv}^w)^T$ can be written as a polynomial in positive roots, with *positive* coefficients.

- $(c_{uv}^w)^T(0) = c_{uv}^w$

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Another generalization of c_{uv}^{w}

Equivariant K-theory! There are several prominent bases that people love. And the notion of positivity changes accordingly.

What is K-theory about?

Suppose V and W are vector spaces over $\mathbb{C}$.

- We "add" vector spaces: $V \oplus W$. The dimensions add.
- We "multiply" vector spaces: $V \otimes W$. The dimensions multiply.
- The 0-vector space is an additive identity. The one-dimensional vector space is a multiplicative identity.
- By formally introducing "subtraction," the set of vector spaces (up to isomorphism) form a ring, denoted $K(pt)$.

Up to isomorphism, V and W are characterized by their dimension. It follows that

$$K(pt) \cong \mathbb{Z}.$$

For an honest vector space, the correspondence is $V \mapsto \dim V$.

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Suppose V and W are vector bundles over a smooth algebraic variety X.

- We can form the direct sum of vector bundles: $V \oplus W$.
- The 0-vector bundle is an additive identity. The trivial one-dimensional vector bundle is a multiplicative identity
- **The problem is cancelation.** we want to have additive inverses. And in particular, if $V \oplus W \cong Z$, then a correct definition for $-V$ must imply that $W \cong Z - V$. However, $V \oplus W \cong Z$ and $V \oplus W' \cong Z$ does not imply that $W \cong W'$.
- Instead, we define $[V]$ as an equivalence class of stably isomorphic vector bundles of the same dimension, i.e.

$$V \sim W \iff V \oplus \mathbb{C}^k \cong W \oplus \mathbb{C}^k \text{ for some } k.$$

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Defining K-theory (topological)

$V \sim W \iff V \oplus \mathbb{C}^k \cong W \oplus \mathbb{C}^k$ for some k. $[V] = \{W | \ W \sim V\}$

- Introduce a formal difference of vector bundles,

$$V - V' \sim W - W' \iff V \oplus W' \cong W \oplus V'.$$

  Note that $V \oplus W \cong V \oplus W'$ is equivalent to $V - W \sim V - W'$, which resolves the problem with cancelation.
- $V - V \sim 0$ (the trivial dimension-0 vector bundle)
- Define $[V] + [W] := [V \oplus W]$. If $[V] = [V']$, then $V \oplus \mathbb{C}^k \cong V' \oplus \mathbb{C}^k$, so $V \oplus \mathbb{C}^k \oplus W \cong V' \oplus \mathbb{C}^k \oplus W$, and $[V \oplus W] = [V' \oplus W]$.
- Similarly, $[V] - [W] := [V - W]$ is well defined. If $[V] = [V']$, then $V \oplus \mathbb{C}^k \cong V' \oplus \mathbb{C}^k$, so $V \oplus \mathbb{C}^k \oplus W \cong V' \oplus \mathbb{C}^k \oplus W$, so that $V - W \sim V' - W$, i.e. $[V - W] = [V' - W]$.
- Form the tensor product of vector bundles: $V \otimes W$, and define $[V] \cdot [W] := [V \otimes W]$. If $V \sim V'$, then $V \otimes W \sim V' \otimes W$.

The set of equivalence classes of vector bundles over X form a ring, denoted $K(X)$.

Defining K-theory (topological)

$V \sim W \iff V \oplus \mathbb{C}^k \cong W \oplus \mathbb{C}^k$ for some k. $[V] = \{W | \ W \sim V\}$

- Introduce a formal difference of vector bundles,

$$V - V' \sim W - W' \iff V \oplus W' \cong W \oplus V'.$$

  Note that $V \oplus W \cong V \oplus W'$ is equivalent to $V - W \sim V - W'$, which resolves the problem with cancelation.
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What is *equivariant* K-theory about?

Suppose a group G acts on vector spaces V and W.

- A vector space V with a group action is a *representation* of G.
- The vector space $V \oplus W$ has an induced group action.
- The vector space $V \otimes W$ has an induced group action.
- The formal difference of vector spaces has an induced group action.

It follows that

$$K_G(pt) \cong R(G),$$

the representation ring of G. A similar story allows us to form a ring $K_G(X)$ out of vector bundles over a smooth algebraic variety X with a G action.

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T- Equivariant K-theory of a point

When $G = T$, an Abelian Lie group (such as S^1, or more generally a product of circles):

- Each finite dimensional representation of T breaks into a direct sum of irreducible representations.
- An irreducible representation of T is *one-dimensional*.
- An irreducible representation of T is characterized by its *weight*, an element of the weight lattice of the dual $\mathfrak{t}^*$ of the Lie algebra $\mathfrak{t}$ to T.

T- Equivariant K-theory of a point

- In the case that $T = S^1$, the weight is simply an integer, indicating how S^1 spins each copy of $\mathbb{C}$. For each $\theta \in S^1$ and $z \in \mathbb{C}$, the action given by $\theta \cdot z = e^{in\theta} z$ is weight n.
- Let t indicate the weight 1 action of S^1 on $\mathbb{C}$. Then

$$K_{S^1}(pt) \cong \mathbb{Z}[t, t^{-1}].$$

- For example, the representation $\mathbb{C} \oplus \mathbb{C}$ of S^1 given by

$$\theta \cdot (z_1, z_2) = (e^{2i\theta} z_1, e^{-3i\theta} z_2)$$

is indicated in the ring by the element $t^2 + t^{-3}$.
- By picking a splitting of $T \cong S^1 \cdots S^1$,

$$K_T(pt) = \mathbb{Z}[t_1, \cdots t_n, t_1^{-1}, \cdots t_n^{-1}],$$

the Laurent polynomials in n-variables.

T- Equivariant K-theory of the flag manifold, G/B

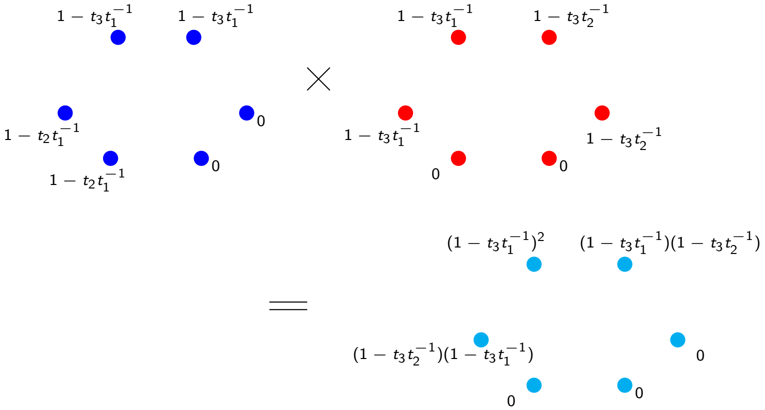
Let $(G/B)^T$ denote the fixed points of T on G/B. The inclusion $(G/B)^T \hookrightarrow G/B$ induces in *injection*

$$\begin{aligned} K_T(G/B) &\hookrightarrow K_T((G/B)^T) \\ &= \bigoplus_W K_T(pt) \\ &= \bigoplus_W Z[t_1, \cdots t_n, t_1^{-1}, \cdots t_n^{-1}]. \end{aligned}$$

There's an algebraic description of the image of this map. For any class $\beta \in K_T(Fl(n, \mathbb{C}))$ and fixed point $w \in (Fl(n, \mathbb{C})^T)$, we write $\beta|_w$ to indicate the restriction of β to the fixed point w.

Why is this so awesome?

We can multiply $K_T(G/B)$ classes by multiplying the corresponding Laurent polynomials at each point!



A geometric story

Theorem

Fact: For each $w \in S_n$, there is a class $\xi_w \in K_T(G/B)$ with $\xi_w|_v \neq 0$ if and only if $v \geq w$ in the Bruhat order.

(Additional properties are required so that these classes are uniquely defined.) These classes are dual to structure sheafs on *Schubert varieties*

$$X^w := \overline{BwB/B}$$

Theorem

Fact: The set of classes $\{\xi_w\}$ over all $w \in S_n$ form a basis of the ring as a module over $K_T(pt)$.

A geometric story

Since the elements $\{\xi_w\}$ form a basis, the coefficients $p_{uv}^w \in K_T(pt)$ are defined by the relationship

$$\xi_u \xi_v = \sum_{w \in S_n} p_{uv}^w \xi_w,$$

then p_{uv}^w have can be written in Laurent polynomials with coefficient that have a predictable sign.
In particular, Bill Graham conjectured that

$$(-1)^{\ell(u)+\ell(v)+\ell(w)} p_{uv}^w \in \mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+}$$

and the statement was subsequently proved by Anderson-Griffeths-Miller.
This phenomenon is called *positivity*.

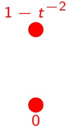
Schubert basis: a simple example

A simplest case: flags on $\mathbb{C}^2$ with an S^1 action on it (weight 2).

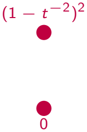
The ξ_w basis:

The class ξ_1

The class $\xi_{[12]}$



In particular, $(\xi_{[12]})^2 =$



$= (1 - t^{-2}) \cdot$

$$\left(\xi_{[12]}\right)^2 = (1 - t^{-2})\xi_{[12]} = -(t^{-2} - 1))\xi_{[21]},$$

meaning

$$p^{[12]}_{[12],[12]} = (-1)^{\ell([21])+\ell([21])+\ell([21])}(t^{-2} - 1),$$

so it satisfies the described positivity.

More on the K-theory classes ξ_u

- We may define basis $\{\xi_v\}_{v\in W}$ is dual to the basis given by $[\mathcal{O}_{X^w}] \in K_T(G/B)$, under a natural pairing (push-forward to a point).
- It turns out that $\xi_v = [\mathcal{O}_{X_v}(\partial X_v)]$, the sheaf of functions on X_v that vanish on the boundary.

Theorem

Fix $v, w \in W$. Let $\mathcal{W}$ be a word with $\prod \mathcal{W} = w$. Then

$$\xi_v|_w = \sum_{\mathcal{V}\subseteq\mathcal{W},\ \tilde{\prod}\mathcal{V}=v} \ \sum_{\mathcal{R}:\mathcal{V}\subseteq\mathcal{R}\subseteq\mathcal{W}} (-1)^{|\mathcal{R}\setminus\mathcal{V}|} \prod_{t\in\mathcal{R}} (1 - e^{(\prod_{j\in\mathcal{W}, j\leq t} r_j)\alpha_t})$$

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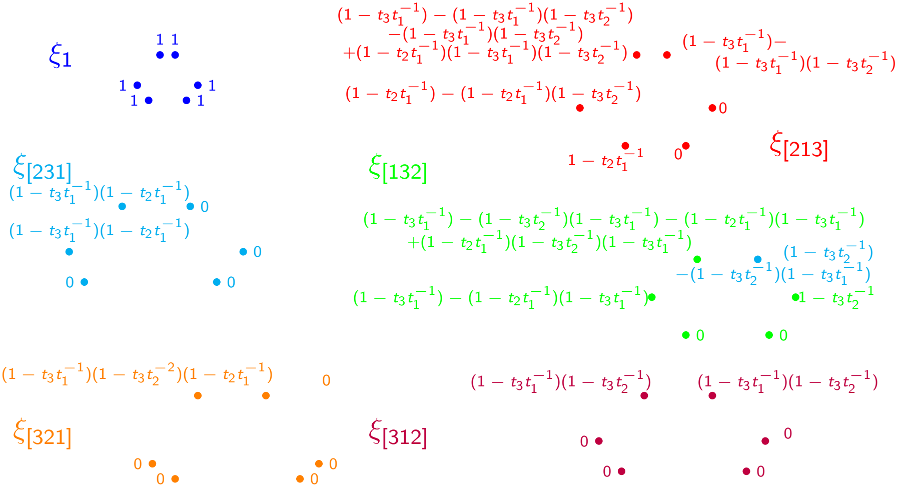
Schubert basis for flags on $\mathbb{C}^3$

Opposite Schubert Varieties, indexed by $w \in S_3$. Here $e^{\alpha_i} = e^{x_i - x_{i+1}} = t_i t_{i+1}^{-1}$



A geometric story

There are no complete positive formulas for the general case.
For a simple reflection $r_j = (j, j+1)$, let δ_j be the K-theoretic isobaric divided difference operator given by

$$\delta_j f := \frac{(f - e^{-\alpha_j} r_j \cdot f)}{(1 - e^{-\alpha_j})}.$$

Theorem (G-Knutson)

Let the equivariant K-theoretic intersection numbers p^w_{uv} be defined by $\xi_v \xi_w = \sum_{u \in W} p^u_{vw} \xi_u$. For any reduced word U for u,

$$p^u_{vw} = \sum_{V \subset U, W \subset U, \widetilde{\prod} V = v, \widetilde{\prod} W = w} \left(\prod_U (\widehat{e^{-\alpha_i}})^{i \in \overline{V \cap W}} (-\delta_i)^{[i \in \bar{V} \cap \bar{W}]} \right.$$
$$\left. (\widehat{1 - e^{-\alpha_i}})^{[i \in V \cap W]} \, r_i^{[i \in V \cup W]} \right) \cdot 1,$$

More on coefficients

$$p_{vw}^u = \sum_{V\subset U, W\subset U, \tilde{\prod} V=v, \tilde{\prod} W=w} \left(\prod_U (\widehat{e^{-\alpha_i}})^{i\in\overline{V\cap W)}} (-\delta_i)^{[i\in \bar{V}\cap\bar{W}]} \right.$$
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- Not a positive formula, not even predictably positive.
- A direct formula (not inductive).
- May lead to positive formulas for specific kinds of permutations (Monk or Pieri).
- Should lead to a recursive formula.

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Analogous statements in equivariant cohomology

For a simple reflection $r_j = (j, j+1)$, let ∂_j be the divided difference operator given by $\partial_j f := \frac{(f - r_j \cdot f)}{\alpha_j}$.

Theorem (G-Knutson)

For any reduced word W for w,

$$c_{uv}^{w} = \sum_{U \subset W, U \subset W, \prod V = v, \prod U = u} \left(\prod W \partial_i^{[i \in \bar{U} \cap \bar{V}]} \widehat{\alpha_i}^{[i \in U \cap V]} r_i \right) \cdot 1,$$

where the sum is over reduced words U and V.

Corollary

Let $\overline{s} = r_\alpha s$ for any s. If $\overline{w} < w$,

$$c_{u,v}^{w} = \partial_\alpha r_\alpha \hat{c}_{u,v}^{\overline{w}} \cdot 1 + [\overline{u} < u] c_{\overline{u},v}^{\overline{w}} + [\overline{v} < v] c_{u,\overline{v}}^{\overline{w}} + [\overline{u} < u][\overline{v} < v] \alpha c_{\overline{u},\overline{v}}^{\overline{w}}$$

Returning to our original intersection theorem

The original question was how many lines intersect 4 given lines. We determined it is the triple intersection

$$[X_1] \cap [X_2] \cap [X_3] \cap [X_4].$$

In equivariant cohomology, this is represented by a polytope decorated by "vectors" that indicate equivariant cohomology classes.

- The polytope we associate to X_1 is the convex hull of fixed points in a T action on $Gr(2,4)$, living in the Lie algebra dual $\mathfrak{t}^*$ of T.
- We naturally associate to each such polytope a cohomology class in H_T^* to each fixed point. The degree of each polynomial is the codimension of the polytope.
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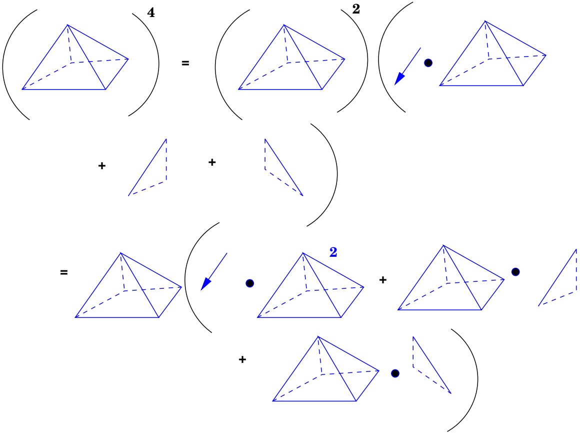
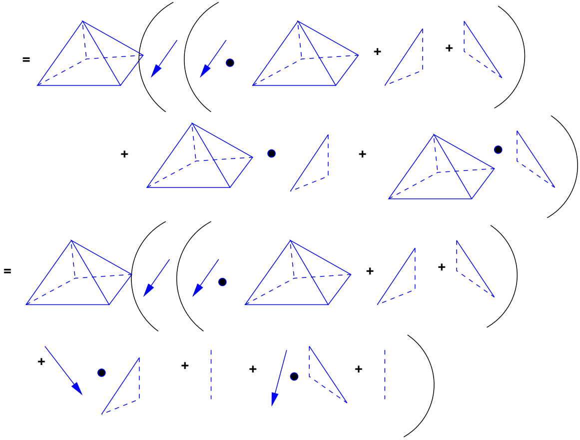
Intersection solution using graphs

Here is a flavor:



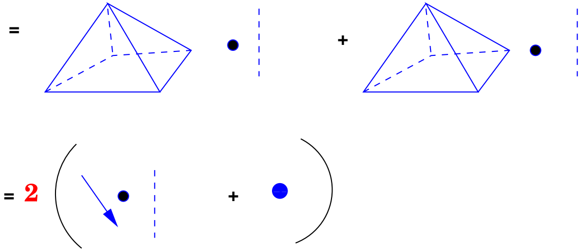
Intersection solution using graphs

The we simplify using the algebra (which is hidden)



Intersection solution using graphs

Now we evaluate the polynomial at 0, which means we kill all terms with arrows.



when we kill the last term with an arrow, we get two (times a point).