L^2-norm preserving nonlinear heat flows

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In [*An optimal partition problem for eigenvalues*, Journal of scientific Computing 31 (2007), 1-2, pp. 5–18], Caffarelli and Lin addressed the problem to look for a bilinear control $m(t) \in \mathbb{R}$ which keeps the L^2 norm of the heat flow on a bounded domain $\Omega \subset \mathbb{R}^d$

$$\begin{cases} \partial_t u = \Delta u + m(t)u, \\ u_{|\partial\Omega} = 0, \\ u(0) = u_0 \in H_0^1(\Omega), \end{cases} \tag{1}$$

constant in time. The solution consists of taking a feedback of the form

$$m(t) = \mu[u(t)]$$

for a suitable nonlocal operator μ. This problem has therefore clear connections with domain invariance for nonlinear diffusions—a topic that has been widely investigated over the years. This talk will discuss such connections and then move on to similar issues for the nonlinear equation

$$\begin{cases} \partial_t u = \Delta u + g|u|^{2\sigma} u + m(t)u, \\ u_{|\partial\Omega} = 0, \\ u(0) = u_0 \in H_0^1(\Omega), \end{cases} \tag{2}$$

where $g \in \mathbb{R}$ and $\sigma > 0$. Problem (2) was studied by Ma and Cheng in [*Nonlocal heat flows and gradient estimates on closed manifolds*, Journal of Evolution Equations 9 (2009), 4, pp. 787–807] in the case of $g < 0$, where the associated energy functional is nonnegative. In a recent joint work with P. Antonelli and B. Shakarov, we have extended the above results by considering $g \in \mathbb{R}$ and the case when either Ω is a bounded domain or $\Omega = \mathbb{R}^d$. Note that the positive sign of g implies that the energy does not control the H^1-norm of the solution in general: it is the L^2 constraint provided by the dynamics (2) that prevents possible finite-time blow-up. Constancy of the energy has also interesting consequences on the asymptotic behaviour of solutions. In general, we obtain weak convergence in H^1 to a stationary state. For a ball, we can prove strong asymptotic convergence to the ground state when the initial condition is positive.