Motivation

First Consider a PDE model:

$$
\left.\begin{aligned}
-\nabla\cdot(e^m \nabla u) &= f && \text{in } \Omega \\
e^m \nabla u\cdot n &= 0 && \text{on } \Gamma_1 \cup \Gamma_3 \\
e^m \nabla u\cdot n &= \beta(T_m - u) && \text{on } \Gamma_2 \\
e^m \nabla u\cdot n &= s && \text{on } \Gamma_4
\end{aligned}\right\}
$$

models heat flow across a conductive surface

Inverse Problems: There is some unknown parameter of interest that we seek to infer using knowledge of our governing model and data measurements y.

- inversion parameter: $m(x)$ - spacially varying log-conductivity
- auxiliary parameters: β - heat transfer coef. of medium
  $s(x)$ - boundary source term
  $f(x)$ - heat source in domain
- experimental parameters: y - data measurements
- complementary parameters: auxiliary + experimental parameters

Frequentist Perspective: Solve inverse problem to obtain an estimate of the inversion parameter m^*.

Bayesian Perspective: Solve inverse problem to obtain a posterior distribution of the inversion parameter.

HDSA: We use derivative based sensitivity analysis to determine the sensitivity of the solution of the inverse problem to perturbations of the complementary parameters.

- informs experimental priorities
  - sensor design
  - estimating auxiliary parameters
- dimension reduction for OEDUU
- insight into the inverse problem

HDSA for Bayesian Nonlinear Inverse Problems

Data Model: $y = f(m, \theta) + \eta$

- f : param-to-observable map
- θ : complementary params
- η : additive Gaussian noise, $\eta \sim N(0, \Gamma_n)$

Data likelihood: $\pi_{like}(y|m) \propto \exp\left(-\frac{1}{2}(f(m,\theta)-y)^T \Gamma_n^{-1}(f(m,\theta)-y)\right)$

- distribution of data y, given m

Prior Distribution: $\mu_{pr} = N(m_{pr}, C_{pr})$

- prior knowledge of m

MAP point:

$$J(m,\theta) = \underbrace{\frac{1}{2}\langle Qu - y, \Gamma_n^{-1}(Qu-y)\rangle}_{} + \underbrace{\frac{1}{2}\langle A(m-m_{pr}), A(m-m_{pr})\rangle}_{}, \quad A = \Gamma_{pr}^{-1/2}$$

$$m_{MAP} = \underset{m}{\operatorname{argmin}}\ J(m,\theta)$$

Sensitivity of the MAP point:

apply implicit function theorem to J_m

results in a continuously differentiable mapping

$$\mathcal{F} : N(\theta^*) \to N(m^*).$$

We define our sensitivity operator

$$D = \mathcal{F}_\theta(\theta^*) = -H^{-1}B \quad \text{where}$$

$$H = J_{m,m}(m^*, \theta^*), \quad B = J_{m,\theta}(m^*, \theta^*)$$

We interpret $D\bar{\theta}$ as the sensitivity of the MAP point when the comp. params are perturbed in the direction $\bar{\theta}$.

Pointwise Sensitivity Indices: compare within parameters

$$S_k^i = \frac{\|D e_k^i\|_M}{\|e_k^i\|_\Theta}$$

Generalized Sensitivity Indices: compare across parameters

$$S_k = \max_{\theta} \frac{\|D T_k \theta\|_M}{\|\theta\|_\Theta}$$

Measures of Posterior Uncertainty

Bayes Risk: approximate by sample averaging

$$\hat{\Psi}_R(\theta) = \frac{1}{n_s} \sum_{i=1}^{n_s} \|m_{MAP}(y_i, \theta) - m_i\|^2 \quad \text{where}$$

$$y_i = f(m_i, \theta) + \eta_i \, , \quad m_i \text{ are prior draws}$$

$$\hat{\Psi}_{risk}(\theta) = \frac{1}{n_s} \sum_{i=1}^{n_s} \langle m_{MAP}(y_i,\theta), m_{MAP}(y_i,\theta)\rangle - 2\langle m_{MAP}(y_i,\theta), m_i\rangle + \langle m_i, m_i\rangle$$

$$\frac{d}{d\theta_j}\hat{\Psi}_{risk}(\theta^*) = \frac{1}{n_s}\sum_{i=1}^{n_s} 2\left\langle \frac{d}{d\theta_j} m_{MAP}(y_i,\theta^*), m_{MAP}(y_i,\theta^*)\right\rangle - 2\left\langle \frac{d}{d\theta_j} m_{MAP}(y_i,\theta^*), m_i\right\rangle$$

$$= \frac{1}{n_s}\sum_{i=1}^{n_s} 2\langle D_j^i, m_{MAP}(y_i,\theta^*)\rangle - 2\langle D_j^i, m_i\rangle \, , \quad j = 1, 2, \ldots, p$$

* note that D_j^i is the j^{th} column of sens. op. D^i which depends on the i^{th} sample draw.

let $D_j^R = \frac{d}{d\theta_j}\hat{\Psi}_{risk}(\theta^*)$ and $D^R = \begin{bmatrix} D_1^R \\ \vdots \\ D_p^R \end{bmatrix} = \nabla_\theta \hat{\Psi}_R(\theta^*)$, then $D^R \bar{\theta}$ can be interpreted as the sens. of $\hat{\Psi}_R$ wrt a perturbation of the params in the direction $\bar{\theta}$.

* Note that building D^R requires building n_s sens. op. $D^i = -H^{-1}B$, i.e. many Hessian solves.